SCATTERING OF A PLANE WAVE BY A SINGULAR POTENTIAL

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 517.946.4

MATHEMATICS

A. A. ARSEN' EV

SCATTERING OF A PLANE WAVE BY A SINGULAR POTENTIAL

(Presented by Academician A. N. Tikhonov on 28 IV 1967)

Numerous works have been devoted to quantum scattering theory in the stationary formulation; closest to us in subject matter are the works ([1,2]). In all the works the potential $V(x)$ was assumed to be an integrable function, since otherwise the Lippmann–Schwinger equation, which served as the principal tool of investigation in ([1,2]), has an essential singularity (some special cases of singularities were studied by L. G. Mikhailov ([3])). However, from the physical point of view the assumption of integrability of the potential $V(x)$ is too restrictive: there exist problems in which the potential $V(x)$ becomes $+\infty$ on a set of positive measure. We propose such an integral equation whose kernel is a bounded integrable function independently of the dimension of the space N and of the singularity of the potential $V(x)$.

Let R_N be N-dimensional Euclidean space ($N \geqslant 3$); Δ the Laplace operator; $V(x)$ a scalar function (potential), $0 \leqslant V(x) \leqslant \infty$; $\Omega = \{x; V(x) = +\infty\}$. $\widetilde{H}$ is the operator defined on functions finite in $R_N \setminus \Omega$ by the formula $\widetilde{H}u = -\Delta u + V(x)u$.

Put $V_M(x) = \min\{V(x), M\}$ $(M \geqslant 0)$, and in what follows we shall mark by the index M all quantities referring to the potential $V_M(x)$.

Below we assume everywhere that the potential $V(x)$ satisfies the following two conditions: 1) every function $V_M(x)$ is nonnegative and satisfies the Hölder condition (locally); 2) there exist constants $a > 0$, $C < \infty$, $R < \infty$, independent of M, such that for all x lying outside the ball of radius R, the estimate $V(x) \leqslant C|X|^{-N-a}$ holds.

Consider the Cauchy problem for the heat equation (in the whole space):

$$u(x,0) = u_0(x), \qquad \partial u/\partial t = -\widetilde{H}_M u, \qquad x \in R_N, \quad t > 0, \quad u \in L^\infty, \qquad (1)$$

and let $G_M(x,y,t)$ be the Green function of problem (1). Represent it in the form:

$$G_M(x,y,t) = G_0(x,y,t) - g_M(x,y,t);$$

$$G_0(x,y,t) = (4\pi t)^{-N/2} \exp\{-(x-y)^2/4t\}.$$

Lemma 1. 1. For any $t > 0$ there exists the limit

$$\lim_{M\to\infty} g_M(x,y,t) = g(x,y,t).$$

2. In the uniform operator topology of the space $[L^p \to L^q;\ 1 \leqslant p \leqslant \infty,\ 1 \leqslant q < \infty]$* the operators $\hat{g}_M(t)$** converge to the operator $\hat{g}(t)$.
3. The operator $\hat{g}(t)$ is completely continuous as an operator from $[L^p \to L^q,\ 1 < p \leqslant \infty,\ 1 \leqslant q < \infty]$.

* $[A_0 \to B, \pi]$ is the space of all bounded linear operators from a Banach space A into a Banach space B, where A and B satisfy condition π.

** $\hat{g}(t)$ is the integral operator with kernel $g(x,y,t)$.

Corollary. The operators $\hat{G}_M(t)$ converge to the operator $\hat{G}(t)$; the operator function $\hat{G}(t)$ is a semigroup in L^p, $1 < p < \infty$.

It can be verified that in $L^p(R_N \setminus \Omega)$ the operator function $G(t)$ is a semigroup of class C_0.

Convergence of the operators $\hat{G}_M(t)$ to $\hat{G}(t)$ in the metric $[L^\infty \to L^\infty]$ need not hold; however, $\hat{G}(t)$ is a semigroup in L^∞. Let $\hat{A}$ be the infinitesimal operator of the semigroup $\hat{G}(t)$, $Hu = -\hat{A}u$. The operator H is an extension of the operator $\hat{H}$.

The problem that we have to solve is formulated as follows: to find that solution of the equation

$$Hu = \lambda u, \qquad x \in R_N, \qquad u \in L^\infty, \qquad \lambda > 0, \tag{2}$$

which can be represented in the form

$$u(x,k) = e^{ikx} + \varphi(x,k), \tag{3}$$

where $k^2 = \lambda$, and the function $\varphi(x,k)$, as $|x| \to \infty$, satisfies the estimates

$$\varphi(x,k) = O\left(|x|^{(1-N)/2}\right) \left(\frac{\partial}{\partial |x|} - i|k|\right) \varphi(x,k) o\left(|x|^{(1-N)/2}\right), \quad |x| \to \infty. \tag{4}$$

Consider the equation

$$e^{-\lambda t}u = \hat{G}(t)u. \tag{5}$$

Substituting (3) into (5), we obtain that the function $\varphi(x,k)$ satisfies the equation

$$(e^{-\lambda t} - \hat{G}_0)\varphi = \hat{g}(t)\varphi.$$

Let $K^+(\lambda)$ be the integral operator with kernel

$$K^+(\lambda, r_{xy}) = \left(\frac{1}{2\pi}\right)^{N/2} \lim_{\varepsilon\to+0} \int_0^\infty \frac{e^{-\rho^2 t}}{e^{-(\lambda+i\varepsilon)t} - e^{-\rho^2 t}} \frac{J_{N/2\alpha-1}(r_{xy}\rho)}{(r_{xy})^{N/2\alpha-1}} \rho^{N/2}\, d\rho$$

$$(r_{xy} = |x-y|).$$

Lemma 2. If $\varphi(x) \in L^p \cap L^\infty$, where $p > 2N/(N-1)$, then

$$(e^{-\lambda t} - G_0)e^{\lambda t}(E + K^+(\lambda))\varphi = e^{\lambda t}(E + \hat{K}^+(\lambda))(e^{-\lambda t} - G_0)\varphi = \varphi.$$

Put, by definition,

$$\hat{T}^+(\lambda) = -e^{\lambda t}(E + \hat{K}^+(\lambda))\hat{g}, \qquad \hat{T}^+_M(\lambda) = -e^{\lambda t}(E + \hat{K}^+(\lambda))\hat{g}_M$$

and consider the equations

$$\psi(x,k,\lambda) = (\hat{T}^+(\lambda)(e^{iky} + \psi))(x,k,\lambda), \qquad 0 < \lambda < \infty, \qquad k \in R_N; \tag{6}$$

$$\psi_M(x,k,\lambda) = (\hat{T}^+_M(\lambda)(e^{iky} + \psi_M))(x,k,\lambda), \qquad 0 < \lambda < \infty, \qquad k \in R_N. \tag{7}$$

The number $t > 0$ enters these equations as a parameter.

Let Ω_1 be the largest open connected set contained in the set $R_N \setminus \Omega$ and containing points of the sphere $|x| = 2R$; $\Omega_2 = R_N \setminus (\Omega \cup \Omega_1)$.

The main result of our work is formulated in Theorems 1-3.

Theorem 1. There exists a countable set of points $\{\lambda_i\}$, independent of $t > 0$, having the property that for each λ_i the equation

$$H\varphi = \lambda_i\varphi$$

has m_i, $1 \le m_i < \infty$, nontrivial solutions; all these solutions belong to L^∞, vanish outside the set Ω_2 (whence it follows that if $\operatorname{mes}\Omega_2 = 0$, then the set $\{\lambda_i\}$ is empty) and satisfy the equality $T^+(\lambda_i)\varphi = \varphi$, while for any $\lambda \in (0,\infty)\setminus\{\lambda_i\}$ the operator $(E - T^+(\lambda))^{-1} \in [L^q \to L^q,\ 2N/(N-1) < q \le \infty]$.

Theorem 2. The set of points $\{\lambda_i\}$ discussed in Theorem 1 has the following properties: it is located on the positive ray from the point 0, and this distance is bounded below by a quantity depending only on the measure of the domain Ω_2; moreover, the number of points of the set $\{\lambda_i\}$ lying in the interval $(0,\lambda)$ satisfies the asymptotic estimate

$$N(\lambda) = \sum_{\lambda_i \leqslant \lambda} 1 \leqslant \frac{\operatorname{mes}\Omega_2}{(4\pi)^{N/2}\Gamma(N/2+1)}\lambda^{N/2} + o(\lambda^{N/2}).$$

Theorem 3. If $\lambda \notin \{\lambda_i\}$, then there exists a solution of problem (2)–(4), namely

$$\varphi(x,k) = \psi(x,k,k^2),$$

where ψ is a solution of equation (6). For sufficiently small t, this solution exists, is unique, and is the limit as $M \to \infty$, in the metric L^q, $2N/(N-1) < q < \infty$, of solutions of equation (7).

The author expresses his deep gratitude to A. A. Samarskii and V. A. Il’ in for numerous consultations, and to the participants of A. N. Tikhonov’ s seminar for discussion of the results.

Moscow State University
named after M. V. Lomonosov

Received
6 IV 1967

REFERENCES

1. Teruo Ikebe, Arch. Rat. Mec. and Analysis, 5, No. 1, 1 (1960).

2. A. Ya. Povzner, Matem. sborn., 32, (74), 1, 109 (1953).

3. L. G. Mikhailov, Tr. AN TadzhSSR, Department of Phys. and Math., 1, Dushanbe, 1963.

Note: Figure translations are in progress. See original paper for figures.