ON THE FUNDAMENTAL THEOREM OF HOMOLOGICAL DIMENSION THEORY

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Abstract

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ON THE FUNDAMENTAL THEOREM OF HOMOLOGICAL DIMENSION THEORY

1. In this note the fundamental theorem of homological dimension theory—in a form directly adjoining the one originally given by me in 1930 (see ([1]))—is proved for bicompacta (and consequently also for arbitrary normal spaces)*. The novelty consists in the fact that, with maximum generality, the formulation proposed here is extremely simple and transparent: I use only the concepts of a cycle and of homology on a finite simplicial complex and dispense with any inverse limits whatever. The proof rests only on the elementary case of Hopf' s theorem on mappings of an n-dimensional polyhedron onto an n-dimensional complex (see, for example, ([2]), p. 70).

2. By a covering in this paper is always meant a finite open covering. The nerve of a covering ω is denoted by N_ω. If $\Phi \subset X$, then $N_{\omega\Phi}$ denotes the subcomplex of the nerve N_ω consisting of all simplices all vertices of each of which correspond to elements of the covering ω having common points with the set Φ. By the dimension of a space X we shall always mean the dimension $\dim X$, defined by means of coverings.

The following is proved.

Fundamental theorem (Theorem 1). *If a bicompactum X has (finite) dimension $\dim X = n > 0$, then there exist a closed set $\Phi \subset X$ and a covering ω of the space X such that for every covering ω' inscribed in ω the following assertions may be made:*

(a) *On the nerve $N_{\omega'}$ there is an n-dimensional relative cycle $z^n_{\omega'}$ mod $N_{\omega'\Phi}$ with respect to some (depending on ω') modulus $m = m_{\omega'}$, whose projection $\delta^{\omega'}_{\omega} z^n_{\omega'}$ into the nerve N_ω is not homologous to zero on the complex N_ω,* mod $N_{\omega\Phi}$.

(b) *On the subcomplex $N_{\omega'\Phi}$ of the nerve $N_{\omega'}$ there is an $(n-1)$-dimensional cycle $z^{n-1}_{\omega'}$ modulo $m_{\omega'}$, homologous to zero on $N_{\omega'}$, and possessing the property that its projection $\delta^{\omega'}_{\omega} z^{n-1}_{\omega'}$ is a cycle (of the complex $N_{\omega\Phi}$) not homologous to zero on $N_{\omega\Phi}$.***

Finally:

(c) *For $r > n$, for every covering ω of the space X there is a covering ω' inscribed in ω (of the space X) such that, for any closed $\Phi \subset X$, every r-dimensional relative cycle $z^r_{\omega'}$ on $N_{\omega'}$ mod $N_{\omega'\Phi}$ and every $(r-1)$-dimensional cycle $z^{r-1}_{\omega'}$ on $N_{\omega'\Phi}$, homologous to zero on $N_{\omega'}$, are equal to zero.*

Assertion (c) is obvious: it suffices to take as ω' any covering inscribed in ω of multiplicity $n+1$.

3. We begin the proof of the main part of the theorem with some preliminary remarks.

By $\bar{Q}$ we shall denote once and for all an n-dimensional closed complex with center of gravity c and interior Q, lying in n-dimensional Euclidean space R^n, in turn embedded in some R^{2n+1}. Under

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* The passage from normal spaces X to bicompacta is effected automatically by means of the maximal bicompact extension βX.

** From what has been said it evidently follows that $\delta^{\omega'}_{\omega} z^{n-1}_{\omega'} \sim 0$ on N_ω and, on the other hand, $z^{n-1}_{\omega'}$ is not ~ 0 on $N_{\omega'\Phi}$ (just as z^n_ω is not ~ 0 on N_ω, mod $N_{\omega\Phi}$).

By a "smaller" simplex ($\bar{Q}_0, \bar{Q}_1$, etc.) we shall always mean a simplex with the same center of gravity as, and homothetic to, the basic simplex $\bar{Q}$ (hence lying strictly inside Q).

Let φ be a continuous mapping of a once-and-for-all given n-dimensional bicompactum X into the simplex $\bar{Q}$. Let $\omega = \{O_1, \dots, O_s\}$ be a covering of the bicompactum X of multiplicity $n+1$. If $\varepsilon > 0$ is given, then by N_ω we shall denote the nerve of the covering ω, realized in the following way as a triangulation lying in $R^{2n+1} \supset R^n \supset \bar{Q}$. For each $i = 1, 2, \dots, s$ choose a point $p_i \in O_i$, and, moreover, if a closed set $\Phi \subset X$ has been distinguished, then for all $O_i \in \omega$ with $O_i \cap \Phi \neq \Lambda$ choose $p_i \in O_i \cap \Phi$. We choose the vertices $e_1, \dots, e_s$ of the nerve N_ω in R^{2n+1} in general position so that

$$\rho(e_i, \varphi p_i) < \varepsilon$$

(for $i = 1, 2, \dots, s$). A realization of the nerve N_ω satisfying these conditions will be called canonical (relative to the mapping φ and the number ε).

Assigning to each vertex e_i the point φp_i, we obtain a simplicial mapping g_ω of the nerve N_ω into $\bar{Q}$, which we shall call the canonical ε-shift (of the complex N_ω).

If the covering ω is normal (i.e. consists of open F_σ-sets), then by μ_ω we denote the barycentric mapping of the bicompactum X into the body $\widetilde{N}_\omega$ of the nerve N_ω.

4. Lemmas.

Lemma 1. *Let a mapping $\varphi \to \bar{Q}$ and a number $\varepsilon > 0$ be given. Then there exist an $\varepsilon' > 0$ and a covering $\omega = \{O_1, \dots, O_s\}$ of the bicompactum X such that, for any canonical realization of N_ω (relative to φ and ε') and the corresponding canonical ε'-shift $g_\omega : N_\omega \to \bar{Q}$, for every normal covering ω' inscribed in ω, the mapping $f = g_\omega \delta_\omega^{\omega'} \mu_{\omega'} : X \to \bar{Q}$ satisfies the inequality*

$$\rho(\varphi x, fx) < \varepsilon \quad \text{for all } x \in X.$$

Indeed, take $\varepsilon' < \varepsilon/3$ and open sets $V_1, \dots, V_\nu$ in the space R^n, covering the simplex $\bar{Q}$ and having diameters $< \varepsilon'$. Put $U_j = \varphi^{-1} V_j$ for $j = 1, 2, \dots, \nu$, and take a covering $\omega = \{O_1, \dots, O_s\}$ of the bicompactum X, of multiplicity $n+1$, inscribed in the covering $\{U_1, \dots, U_\nu\}$. Inscribe in ω any normal covering $\omega' = \{O'_1, \dots, O'_{s'}\}$ of multiplicity $n+1$.

Let the nerve N_ω be canonically realized at the vertices $e_1, \dots, e_s$, and the nerve $N_{\omega'}$ at the vertices $e'_1, \dots, e'_{s'}$ (both realizations are relative to the mapping φ and the number ε').

Take an arbitrary point $x \in X$; suppose it is contained in

$$O'_{k_0}, \dots, O'_{k_r}$$

and only in these elements of the covering ω'. Put

$$e_{i_\lambda} = \delta_\omega^{\omega'} e'_{k_\lambda}, \quad \lambda = 0, \dots, r.$$

Then

$$\mu_{\omega'} x \in |e'_{k_0} \cdots e'_{k_r}| \in N_{\omega'}, \qquad \delta_\omega^{\omega'} |e'_{k_0} \cdots e'_{k_r}| = |e_{i_0} \cdots e_{i_r}|,$$

so that

$$\delta_\omega^{\omega'} \mu_{\omega'} x \in |e_{i_0} \cdots e_{i_r}|.$$

Further,

$$x \in O'_{k_\lambda} \subset O_{i_\lambda} \subset U_{j_\lambda}.$$

Hence

$$\rho(\varphi x, \varphi p_{i_\lambda}) < \varepsilon', \qquad \rho(\varphi x, e_{i_\lambda}) < 2\varepsilon'$$

(all this for $\lambda = 0, \dots, r$). Since

$$\delta_\omega^{\omega'} \mu_{\omega'} x \in |e_{i_0} \cdots e_{i_r}|,$$

we have

$$\rho(\varphi x, \delta_\omega^{\omega'} \mu_{\omega'} x) < 2\varepsilon',$$

and therefore

$$\rho(\varphi x, g_\omega \delta_\omega^{\omega'} \mu_{\omega'} x) < 3\varepsilon' < \varepsilon,$$

which proves Lemma 1.

Lemma 2. *Let $\varphi : X \to \bar{Q}$ be an essential mapping. Then, for any smaller simplex $\bar{Q}_0 \subset Q$, one can find a covering $\omega = \{O_1, \dots, O_s\}$ of the bicompactum X and a number $\varepsilon > 0$ such that, for an arbitrary normal covering $\omega' = \{O'_1, \dots, O'_{s'}\}$, of multiplicity $n+1$, inscribed in ω, and for the canonical ε-shift $g_\omega : N_\omega \to \bar{Q}$, the simplicial mapping*

$$\psi = g_\omega \delta_\omega^{\omega'} : \widetilde{N}_{\omega'} \to \bar{Q}$$

essentially covers the simplex $\bar{Q}_0$ (i.e. the mapping of the polyhedron $\psi^{-1}\bar{Q}_0$ onto $\bar{Q}_0$ is essential).

The proof is based on the following slight modification of the main lemma for the theorem on essential mappings (see, for example, ([3]), Ch. 6, p. 217), whose proof we leave to the reader:

Lemma 2_0. *With the notation adopted above, there is an $\varepsilon > 0$ such that every continuous mapping $f : X \to \bar{Q}$ differing from φ by less than ε essentially covers the simplex $\bar{Q}_0$.*

But by Lemma 1 there exists a covering ω of the bicompactum X such that for every ω' inscribed in ω, for sufficiently small ε', and for the corresponding canonical ε'-shift $g_\omega : N_\omega \to \bar{Q}$, we have the inequality

$$\rho(\varphi x,\ g_\omega \tilde{\omega}^{\omega'} \mu_{\omega'} x) < \varepsilon \quad \text{for all } x \in X.$$

Hence the mapping $f = g_\omega \tilde{\omega}^{\omega'} \mu_{\omega'} : X \to \bar{Q}$ essentially covers $\bar{Q}_0$. We show that then also the simplicial mapping $\psi = g_\omega \tilde{\omega}^{\omega'} : \widetilde{N}_{\omega'} \to \bar{Q}$ essentially covers $\bar{Q}_0$. Let $X_0 = f^{-1}\bar{Q}_0 = \mu_{\omega'}^{-1}\psi^{-1}Q_0$.

Put $Y = \psi^{-1}\bar{Q}_0 \subseteq \widetilde{N}_{\omega'}$; then $X_0 = \mu_{\omega'}^{-1}Y$. Denoting by S_0 the boundary of the simplex Q_0, put further $\Psi = \psi^{-1}S_0 \subset Y$. It is necessary to prove that the mapping $\psi : Y \to \bar{Q}_0$ is essential. Otherwise there is a mapping $\psi_1 : Y \to S_0$ for which $\psi_1 y = \psi y$ for every $y \in \Psi$.

We consider the barycentric mapping $\mu_{\omega'} : X_0 \to Y$ and define $\varphi_1 = \psi_1 \mu_{\omega'} : X_0 \to S_0$. For $x \in f^{-1}S_0$ we have $S_0 \ni fx = g_\omega \tilde{\omega}^{\omega'} \mu_{\omega'} x = \psi\mu_{\omega'} x$, i.e. $\mu_{\omega'} x \in \psi^{-1}S_0 = \Psi$, and, consequently, by the definition of ψ_1 we have $\psi_1\mu_{\omega'}x = \psi\mu_{\omega'}x$, while by the definition of φ_1 we have $\varphi_1 x = \psi_1\mu_{\omega'}x = \psi\mu_{\omega'}x = fx$. Thus we have a mapping $\varphi_1 : X_0 \to S_0$ coinciding with f on $f^{-1}S_0$, contrary to the essentiality of the mapping f. Lemma 2 is proved.

5. Proof of assertion (a) of Theorem 1. Keeping the notation of Lemmas 1 and 2, take smaller simplexes $\bar{Q}_1 \subset Q_0$ and $\bar{Q}_2 \subset Q_1$. Put $\Phi = \varphi^{-1}(\bar{Q} \setminus Q_1) \subset X$. Suppose that ε in Lemma 2 is less than half the distance from each smaller simplex to its complement (in R^n) in the larger one. Then

$$f^{-1}S_0 \subseteq \Phi. \tag{1}$$

Otherwise we would have a point $x_0 \in X$ for which $fx_0 \in S_0$, but $\varphi x_0 \in Q_1$, and hence $\rho(\varphi x_0, fx_0) > \varepsilon$.

We derive from formula (1) the formula

$$\psi^{-1} S_0 \subseteq N_{\omega\Phi} \cap Y; \qquad N_{\omega\Phi} \subseteq \psi^{-1}(Q \setminus Q_2). \tag{2}$$

We prove the first inclusion in (2). If* $y \in \psi^{-1} S_0$ and $x \in \mu_{\omega'}^{-1} y$, then $fx = \psi\mu_{\omega'} x \in S_0$, i.e. $x \in f^{-1} S_0 \subseteq \Phi$, and every $O'_k \in \omega'$ containing the point x intersects Φ.

Let x be contained in $O'_{k_0}, \dots, O'_{k_r}$, and only in these elements of the covering ω'. Then $y = \mu_{\omega'} x \in |e_{k_0} \dots e_{k_r}| \in N_{\omega\Phi}$. Thus, $\psi^{-1} S_0 \subseteq N_{\omega\Phi}$. Since, moreover, $\psi^{-1} S_0 \subseteq Y$, the first inclusion in formula (2) is proved.

We prove the second inclusion. Let $y \in |e_{k_0} \dots e_{k_r}| \in N_{\omega\Phi}$. We must reduce to a contradiction the inclusion $\psi y \in Q_2$. For this we take some $x \in \mu_{\omega'}^{-1} y$. The inclusion $\psi y \in Q_2$ means $fx \in Q_2$. Since $\rho(\varphi x, fx) < \varepsilon$, it follows that for every $O'_k \subset \omega'$ containing x, we have (recalling that $\operatorname{diam} \varphi O'_k < \varepsilon$) the inclusions

$$\varphi O'_k \subset O(\varphi x, \varepsilon) \subseteq O(fx, 2\varepsilon) \subseteq Q_1,$$

and hence $O'_k \subseteq \varphi^{-1} Q_1$. On the other hand, the inclusion $y \in |e_{k_0} \dots e_{k_r}| \in N_{\omega\Phi}$ means that x is contained in $O'_{k_0}, \dots, O'_{k_r}$ and only in these elements of the covering ω', and that each of these $O'_{k_0}, \dots, O'_{k_r}$ intersects $\Phi = \varphi^{-1}(\bar{Q} \setminus Q_1)$. The required contradiction has been obtained.

* We may always assume (replacing, if necessary, the covering ω' by an inscribed covering subordinate to it) that $\mu_{\omega'} : X \to N_{\omega'}$ is a mapping onto $N_{\omega'}$. In this case ω' is called irreducible if there is no covering inscribed in it whose nerve is a proper subcomplex of the nerve $N_{\omega'}$.

Remark. Let us prove the formula

$$g_\omega \widetilde{N}_{\omega\Phi} \subseteq \overline{Q} \setminus Q_2. \tag{3_ω}$$

Indeed, the inclusion (3_ω) follows from the fact that all vertices of the complex $g_\omega N_{\omega\Phi}$ are points of the form φp_i, where $p_i \in O_i \cap \Phi$; that the simplices of this complex have diameter $< 2\varepsilon$, and that $\varphi\Phi \subseteq \overline{Q} \setminus Q_1$.

Let us sum up: we have an essential mapping $\psi = g_\omega \tilde{\mathfrak{d}}_\omega^{\omega'}$ of the polyhedron Y (of dimension n) onto the n-dimensional complex $\overline{Q}_0$, under which $\psi^{-1} S_0$ is contained in the polyhedron $\widetilde{N}_{\omega\Phi} \cap Y = \Pi_0 \subseteq \psi(\overline{Q}_0 \setminus Q_2)$. Denote now by $N_{\omega'}^{(1)}$ such a subdivision of the triangulation $\widetilde{N}_{\omega'}$ that the polyhedron Y is the body $\widetilde{K}$ of some complex $K \subseteq N_{\omega'}^{(1)}$, and the polyhedron Π_0 is the body of some complex $K_0 \subseteq K$.

We are in the conditions of the already mentioned theorem of Hopf (([2]), p. 70), by virtue of which on the complex K there is a relative cycle $z^n_{\omega'_1}$ mod K_0, with respect to some modulus $m = m_{\omega'}$, which covers under the mapping $\psi : Y \to \overline{Q}_0$ the point c with degree $\mathrm{gr}_c\,\psi = \gamma \neq 0$. Under the canonical shift $\sigma : N^{(1)}_{\omega'} \to N_{\omega'}$ we have $\sigma K_0 \subseteq N_{\omega'\Phi}$, and the relative cycle $z^n_{\omega'_1}$ mod K_0 passes into the relative cycle $\sigma z^n_{\omega'_1} = z^n_{\omega'}$ of the complex $N_{\omega'}$, mod$N_{\omega'\Phi}$, while the boundary $z^{n-1}_{\omega'_1} = \Delta z^n_{\omega'_1}$ passes (remaining throughout in the polyhedron $\widetilde{N}_{\omega'\Phi} \subseteq \overline{Q} \setminus Q_2$) into the cycle $z^{n-1}_{\omega'}$ of the complex $N_{\omega'}$. At the same time the cycle $\psi z^{n-1}_{\omega'_1}$ undergoes a deformation in $\overline{Q}\setminus Q_2$ which does not change the linking coefficient $\nu(c, \psi z^{n-1}_{\omega'_1}) = \mathrm{gr}_c\,\psi z^n_{\omega'_1}$, so that $\mathrm{gr}_c\,\psi z^n_{\omega'} = \mathrm{gr}_c\,\psi z^n_{\omega'_1} = \gamma \neq 0$. Obviously, $\tilde{\mathfrak{d}}^{\omega'}_{\omega} z^n_{\omega'} = z^n_{\omega}$ is a relative cycle of the complex N_ω, mod$\,N_{\omega\Phi}$, with respect to the same modulus $m = m_{\omega'}$, and $\mathrm{gr}_c\,g_\omega z^n_\omega = \mathrm{gr}_c\,g_\omega \tilde{\mathfrak{d}}^{\omega'}_{\omega} z^n_{\omega'} = \mathrm{gr}_c\,\psi z^n_{\omega'} = \gamma \neq 0$. Consequently, z^n_ω is not ~ 0 on N_ω mod $N_{\omega\Phi}$, as was required to prove.

6. Proof of assertion (6) of Theorem 1. We keep our notation. The projection $\mathfrak{d}^{\omega'}_{\omega} z^{n-1}_{\omega'}$ is a cycle of the complex $N_{\omega\Phi}$. Let us prove that it is not homologous to zero on $N_{\omega\Phi}$. Otherwise, let $\mathfrak{d}^{\omega'}_{\omega} z^{n-1}_{\omega'}$ be the boundary of a chain x^n_ω of the complex $N_{\omega\Phi}$. By formula (3_ω), the chain $g_\omega x^n_\omega$ lies in $\overline{Q} \setminus Q_2$, so that $\mathrm{gr}_c\,g_\omega x^n_\omega = 0$; therefore for the cycle $y^n_\omega = z^n_\omega - x^n_\omega$ we have $\mathrm{gr}_c\,g_\omega y^n_\omega = \gamma \neq 0$, which is obviously impossible, since the cycle $g_\omega y^n_\omega$, like every n-dimensional cycle on the n-dimensional complex $\overline{Q}_0$, is homologous in $\overline{Q}_0$ to zero. Theorem 1 is proved.

In the metric case Theorem 1 assumes, as is easy to establish, the following form:

Theorem 2. *For every compactum X of finite dimension $n \geq 1$ there exist a closed set $\Phi \subset X$ and an $\varepsilon > 0$ such that for every (arbitrarily small) $\varepsilon' > 0$ there is, first, an n-dimensional relative ε'-cycle on X* mod Φ, *with respect to some (depending on ε') modulus $m_{\varepsilon'}$, not ε-homologous to zero on X* mod Φ, *and, secondly, on Φ there is an $(n-1)$-dimensional ε'-cycle $z^{n-1}_{\varepsilon'}$ modulo $m_{\varepsilon'}$, ε'-homologous to zero on X, but not even ε-homologous to zero on Φ.*

Here n is the greatest natural number for which at least one of these assertions holds.

The formulation of Theorem 2 (entirely in the spirit of Brouwer's classical works) is in an obvious way equivalent to my main theorem in its original formulation (paper ([1]), p. 195).

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Note: Figure translations are in progress. See original paper for figures.