ON THE GAME ENCOUNTER OF MOTIONS

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Abstract

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Corresponding Member of the Academy of Sciences of the USSR N. N. Krasovskii

ON THE GAME ENCOUNTER OF MOTIONS

Consider the problem ([1–5]) of the game encounter of the pursuing ($y[t]$) and pursued ($z[t]$) motions

$$\dot{y} = Ay + Bu, \qquad \dot{z} = Az + Bv, \tag{1}$$

where the realizations $u[t]$ and $v[t]$ of the controls u and v are constrained by

$$\|u\| \leq \mu, \qquad \|v\| \leq \nu \qquad (\mu > \nu), \tag{2}$$

the symbol $\|q\|$ denoting the Euclidean norm of the vector q. Here y, z are n-vectors; u, v are r-vectors. The vectors under consideration are treated as column vectors; the superscript $'$ will denote transposition, and the symbol $\{q\}_m$ will denote the vector composed of the first m components of the vector q. The process begins at a prescribed time $t = t_0$. The encounter of the motions y and z at the instant $t^* = t_0 + T$ is defined as a situation satisfying the condition

$$\{y[t_0 + T] - z[t_0 + T]\}_m = 0, \tag{3}$$

where the number m is given.

The solution of the minimax problem of the time T until the encounter of the motions y and z is determined by the theorems from works ([2],[3],[6],[7]), if, among the arguments of the function $u = u^0$ specifying the control for the pursuer, the current values of the control $v[t]$ are allowed. Otherwise, the solution of this problem can be obtained in limiting form from a certain approximating scheme ([8]). In this case, again, among the arguments of the function u^0, along with the phase vectors $y[t]$ and $z[t]$, there appears an additional variable $\vartheta(t)$. The limiting motions $y[t], z[t], \vartheta[t]$ generated in this way are included within the framework of generalized solutions ([9]) of the resulting differential equations with discontinuous right-hand sides. This is achieved in the following way.

An extremal control $u = u^0$ is constructed, defining equations (1) as equations in contingencies. For this purpose, in the $(2n + 1)$-dimensional space $\{y, z, \vartheta\}$

($\vartheta > 0$), two subsets W_0 and W^e are distinguished, which are specified by the following conditions: let $\vartheta^0(y, z)$ be the time until the moment of absorption ([5],[10]), $t^0 = t_* + \vartheta^0$, of the process $z[t]$ by the process $y[t]$ (from the state $y[t_*] = y,\ z[t_*] = z$); then the set W_0 is the totality of all states $\{y, z, \vartheta\}$ satisfying the condition $\vartheta \geq \vartheta^0(y, z)$; the complementary set W^e, on the contrary, is the totality of states $\{y, z, \vartheta\}$ that satisfy the condition $\vartheta < \vartheta^0(y, z)$. In the domain W_0, the function $u^0(y, z, \vartheta)$ is chosen nonuniquely and may take any values satisfying the condition

$$\|u^0\| \leq \mu. \tag{4}$$

In the domain W^e, the function $u^0(y, z, \vartheta)$ is defined by the rule of extremal aiming ([5],[8] ,[10]). Let $\varepsilon^0(y, z, \vartheta)$ be the smallest value of ε for which the closed ε-neighborhood $G^{(1)}[y[t], \vartheta; \varepsilon]$ of the reachability domain $G^{(1)}[y[t], \vartheta]$ of the process y (from the state $y[t]$ to the moment $t_\vartheta = t+\vartheta$) contains the reachability domain $G^{(2)}[z[t], \vartheta]$ of the process z (from the state $z[t]$ to the moment t_ϑ). Further, let $u[t] = u_*(y[t], z[t], \vartheta[t])$ be that control which, at the instant t, aims the motion $y[t]$ to a point $q^0 \in G^{(1)}[y[t], \vartheta]$ nearest to that point p^0 at which the boundaries of the domains $G^{(1)}[y[t], \vartheta; \varepsilon^0]$ and $G^{(2)}[z[t], \vartheta]$ touch. If at the point $\{y, z, \vartheta\} \in W^\varepsilon$ the function $u_*(y, z, \vartheta)$ is continuous, then $u^0(y, z, \vartheta) = u_*(y, z, \vartheta)$. In the opposite case $u^0(y, z, \vartheta)$ is again a multivalued function, constrained only by condition (4). The additional variable $\vartheta[t]$ is constrained by the regulating relation

$$(d\vartheta[t]/dt)^+ \leq -1, \tag{5}$$

which is adjoined to the original equations of motion (1) (the symbol $(d\vartheta/dt)^+$ denotes the upper derivative).

By a generalized solution (motion) $\{y[t], z[t], \vartheta[t]\}$ ($t \geq t_0$) of the system (1), (5) for $u = u^0(y, z, \vartheta)$ we shall mean any vector function $\{y[t], z[t], \vartheta[t]\}$ that satisfies the conditions:

1°. The functions $y[t]$ and $z[t]$ are absolutely continuous and for almost all values of t satisfy equations (1), where $u = u^0$ and arbitrary piecewise-continuous realizations $v[t]$ are admissible, constrained by condition (2).

2°. The variable $\vartheta[t]$ is right-continuous for all values of t and satisfies condition (5). In the domain W^ε the function $\vartheta[t]$ is continuous and satisfies the equation $d\vartheta/dt = -1$, while in the domain W_0 the possible values of $\vartheta[t]$ are constrained by the condition $\varepsilon^0(y[t], z[t], \vartheta[t]) = 0$.

The following assertion is valid.

I. *Suppose that for the initial state $y[t_0]$, $z[t_0]$ of system (1) the condition $\vartheta^0(y[t_0], z[t_0]) < \infty$ is fulfilled. Then, for $u = u^0$, every generalized solution $\{y[t], z[t], \vartheta[t]\}$ ($t \geq t_0$) of the system (1), (5) with the initial condition $y[t_0]$,*

$z[t_0]$, $\vartheta[t_0] = \vartheta^0(y[t_0], z[t_0])$ has the property that $\{y[t], z[t], \vartheta(t)\} \in W_0$ for all $t \geq t_0$, until the encounter occurs. This inclusion ensures the encounter of the motions $y[t]$ and $z[t]$ no later than at the moment $t^0 = t_0 + \vartheta^0(y[t_0], z[t_0])$.

Assertion (I) means that

$$\min_u \max_v T_{u,v} = T_{u^0(y,z,\vartheta),\, v^0} = \vartheta^0(y[t_0], z[t_0]) \tag{6}$$

among the controls $u = u(y, z, \vartheta)$, $v = v[t]$ $(v^0[t] = \nu u^0(y[t], z(t), \vartheta[t])/\mu)$, and the given minimax is realized by the control $u = u^0$, which, when $v \neq v^0$, generally forces sliding regimes $y[t]$.

Remark 1. Among the motions $\{y[t], z[t], \vartheta[t]\}$ under consideration, for the pursuer perhaps the most interesting are those for which, for $t \geq t_0$, the equality $\vartheta[t] = \vartheta^0(y[t], z[t])$ holds throughout up to the encounter. Indeed, on these motions, for $t \geq \tau \geq t_0$, the encounter for every τ takes place no later than at the moment $t^0[\tau] = \tau + \vartheta^0(y[\tau], z[\tau])$. From example (7), considered in article (8), it follows that the imposition of the regulating relation (5) is essential for ensuring the encounter of the motions $y[t]$ and $z[t]$.

Thus, the extremal control $u^0(y, z, \vartheta)$ ensures the minimax (6) for the time $T_{u,v}$ until encounter. On the other hand, one can verify by an example that the extremal control $v^0(y, z)$ (or $v^0(y, z, \vartheta)$), selected at each instant of time t from the condition that the motion $z[t]$ be aimed at the moment $t^0 = t + \vartheta^0(y[t], z[t])$ at the point q^0, where the boundaries of the domains $G^{(1)}[y[t], \vartheta]$ and $G^{(2)}[z[t], \vartheta]$ touch, does not ensure the maximin

$$\max_v \min_u T_{u,v} = \vartheta^0(y[t_0], z[t_0]). \tag{7}$$

An example of this kind may be furnished by the problem of the game encounter only in the coordinates of two controllable material points $m^{(1)}$ and $m^{(2)}$, described by the equations

$$m^{(1)}\ddot{\xi}^{(1)} = u, \qquad m^{(2)}\ddot{\xi}^{(2)} = v, \tag{8}$$

where $\xi^{(1)}, \xi^{(2)}$ are three-dimensional vectors, and the controls u and v are constrained by conditions (2). However, the following assertion is valid.

II. For any number $\delta > 0$ one can construct a control $v_\delta[t] = v_\delta(y[t], z[t], \vartheta_\delta[t])$ $(d\vartheta_\delta/dt = -1,\ \vartheta_\delta[t_0] = \vartheta^0(y[t_0], z[t_0]) - \delta$, which preserves the motion $z[t]$ from meeting the motion $y[t]$ for $t_0 \leq t < t_0 + \vartheta^0(y[t_0], z[t_0]) - \delta$, whatever the realization of the control $u[t]$ constrained by condition (2).

The control v_δ again determines the motion $z[t]$ as a generalized solution of the corresponding equation (1). Here the control $v_\delta[t]$ is constructed from the condition of nonincrease of the Lyapunov function

$$\chi(y[t], z[t], \vartheta_\delta[t]) = \int_t^{t+\vartheta_\delta} \varepsilon^0(y[t], z[t], \tau - t)^{-1}\, d\tau$$

with respect to time t. It is then verified that the function $\chi[t] = \chi(y[t], z[t], \vartheta_\delta[t])$ does not increase if the control $v_\delta[t]$ is chosen from the minimum condition

$$\int_t^{t+\vartheta_\delta} \varepsilon^0(y[t], z[t], \tau - t)^{-2}\psi'_\tau(t)B\left(\frac{vB'\psi_\tau(t)}{\|B'\psi_\tau(t)\|} - v_\delta[t]\right) d\tau = \min_{\|v\|\le v}, \tag{9}$$

where $\psi_\tau(t)$ is the vector-function appearing in the conditions of the maximum principle (11) for the problem of determining the quantity $\varepsilon^0(y[t], z[t], \tau - t)$.

Remark 2. The quantity $\vartheta^0(y[t_0], z[t_0])$, generally speaking, is greater than the optimal time ϑ_0 for the problem of program maximin pursuit considered in paper (12). Thus, the use of the realizing values $y[t]$ and $z[t]$ in the law determining the control $v[t]$ by the feedback principle ensures, for the pursued motion, a better result than the program control $v(t)$, which is based only on the initial data $y[t_0]$ and $z[t_0]$. At the same time, the control v_δ described by us, constructed according to the feedback principle, ensures for the pursued object a result $T_\delta = \inf_u T_{u,v}$ arbitrarily close to the best guaranteed result for the pursuer,

$$T^0 = \min_u \max_v T_{u,v}.$$

Above, the problem of exact meeting of the motions y and z in m coordinates was considered. Results I and II, however, are transformed in an understandable way to the case of the problem of γ-meeting, which is defined by the condition

$$\|\{y[t_0 + T] - z[t_0 + T]\}_m\| \le \gamma, \tag{10}$$

where only the time $\vartheta^0(y, z)$ until absorption is replaced by the time $\vartheta^{(\gamma)}(y, z)$ until γ-absorption. It should also be noted that results I and II carry over without substantial changes to the case of arbitrary convex and similar constraints $u \in U$ and $v \in V$ on the controls u and v.

Ural State University
named after A. M. Gorky

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Note: Figure translations are in progress. See original paper for figures.