CHARACTERISTIC COEFFICIENTS OF TWO-DIMENSIONAL DISTRIBUTIONS AND THEIR APPLICATION

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Abstract

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CHARACTERISTIC COEFFICIENTS OF TWO-DIMENSIONAL DISTRIBUTIONS AND THEIR APPLICATION

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1. Consider the class T of continuous strictly monotone one-dimensional distribution functions with total variation equal to one, and two functions $F_1(x)$ and $F_2(y)$ from this class. Let $H(x, y)$ be an arbitrary two-dimensional distribution function, and let $H_1(x) = H(x, \infty)$, $H_2(y) = H(\infty, y)$ be its marginal distribution functions.

Definition. The sequence of complex numbers

$$\lambda_{F_1F_2}(k, s, H) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{2\pi i[k(F_1(x) - {}^1/_2) + s(F_2(y) - {}^1/_2)]\}\, dH(x, y), \tag{1}$$

where $k, s = 0, \pm1, \pm2, ...$, is called the **sequence of two-dimensional characteristic coefficients** (c.c.) of the distribution function $H(x, y)$ with respect to $F_1(x)$ and $F_2(y)$.

Remark. For $k = 1, 2, ...$

$$\lambda_{F_1F_2}(k, 0, H) = \lambda_{F_1}(k, H_1) = \alpha_{F_1}(k, H_1) + i\beta_{F_1}(k, H_1),$$

and for $s = 1, 2, ...$

$$\lambda_{F_1F_2}(0, s, H) = \lambda_{F_2}(s, H_2) = \alpha_{F_2}(s, H_2) + i\beta_{F_2}(s, H_2)$$

are the one-dimensional c.c. of H_1 and H_2, respectively, considered in (1).

From (1) it follows that

$$\lambda_{F_1F_2}(k, s, H) = a_{ks} - d_{ks} + i(b_{ks} + c_{ks}), \tag{2}$$

where

$$a_{ks} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos kz_1 \cos sz_2 \, dH(x,y), \qquad b_{ks} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin kz_1 \cos sz_2 \, dH(x,y), \tag{3}$$

$$c_{ks} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos kz_1 \sin sz_2 \, dH(x,y), \qquad d_{ks} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin kz_1 \sin sz_2 \, dH(x,y),$$

$$k, s = 0, 1, 2, \dots,$$

$$z_1 = 2\pi(F_1(x) - {}^1/_2), \qquad z_2 = 2\pi(F_2(y) - {}^1/_2). \tag{4}$$

Specifying the c.c. (3) is equivalent to specifying the c.c. (1). If x and y are independent, then

$$\lambda_{F_1F_2}(k, s, H) = \lambda_{F_1}(k, H_1)\lambda_{F_2}(s, H_2);$$

the converse assertion is also true, so that the factorization of all two-dimensional c.c. into products of one-dimensional c.c. is a necessary and sufficient condition for independence.

2. Let us express $H(x,y)$ in terms of the c.c. (3). Expand the difference

$$H\left[F_1^{-1}\left(\frac{z_1+\pi}{2\pi}\right), F_2^{-1}\left(\frac{z_2+\pi}{2\pi}\right)\right] - \frac{z_1+\pi}{2\pi}\frac{z_2+\pi}{2\pi} \tag{5}$$

in a double Fourier series in the square $[-\pi \le z_1, z_2 \le \pi]$; returning to the old variables, as was done in $({}^1)$, we obtain

$$H(x,y) = F_1(x)F_2(y) + \Delta(x,y) + \lambda_{00} + \frac{1}{\pi^2}\sum_{k,s=1}^{\infty}\frac{1}{ks}(a_{ks}\sin kz_1 \sin sz_2 +$$

$$+ d_{ks}\cos kz_1 \cos sz_2 - b_{ks}\cos kz_1 \sin sz_2 - c_{ks}\sin kz_1 \cos sz_2) +$$

$$+ \frac{1}{\pi^2}\sum_{k,s=1}^{\infty}\frac{1}{ks}((-1)^k b_{ks}\sin sz_2 + (-1)^s c_{ks}\sin kz_1$$

$$- (-1)^k d_{ks}\cos sz_2 - (-1)^s d_{ks}\cos kz_1), \tag{6}$$

where

$$\Delta(x,y) = [H_1(x) - F_1(x)]F_2(y) + [H_2(y) - F_2(y)]F_1(x), \tag{7}$$

$$\lambda_{00} = \frac{1}{\pi^2}\sum_{k,s=1}^{\infty}\frac{(-1)^{k+s}}{ks}d_{ks} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[F_1(x) - {}^1\!/_2]\,[F_2(y) - {}^1\!/_2]\,dH(x,y).$$

Remark. To obtain from (6) the expansions of the marginal functions $H_1(x)$ and $H_2(y)$, in $\Delta(x,y)$ the differences $H_1(x) - F_1(x)$ and $H_2(y) - F_2(y)$ must be replaced by the corresponding expansions with one-dimensional characteristic coefficients (see (1)).

If at points of discontinuity the symmetry conditions

$$\begin{aligned} F_1(x) &= 1 - F_1(-x), & F_2(y) &= 1 - F_2(-y), \\ H(x,y) &= H_2(y) - H(-x,y), & H(x,y) &= H_1(x) - H(x,-y), \end{aligned} \tag{8}$$

are satisfied, then the characteristic coefficients (1) are real and (6) is simplified to

$$H(x,y) = F_1(x)F_2(y) + \Delta(x,y) + \frac{1}{\pi^2}\sum_{k,s=1}^{\infty}\frac{\lambda_{F_1F_2}(k,s,H)}{ks}\sin kz_1 \sin sz_2. \tag{6'}$$

If, moreover,

$$H_1(x) = F_1(x), \qquad H_2(y) = F_2(y), \tag{8'}$$

then

$$H(x,y) = F_1(x)F_2(y) + \frac{1}{\pi^2}\sum_{k,s=1}^{\infty}\frac{\lambda_{F_1F_2}(k,s,H)}{ks}\sin kz_1 \sin sz_2. \tag{6''}$$

The series (6)–(6″) converge to $H(x,y)$ at all points of continuity; on the other hand, they converge in the mean, since the integral

$$\rho^2(H;F_1F_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[H(x,y) - F_1(x)F_2(y)]^2\,dF_1(x)\,dF_2(y), \tag{9}$$

which we shall call the squared distance from $H(x,y)$ to $F_1(x)F_2(y)$, exists for any distribution functions.

From (6)–(6″) the following assertions follow immediately:

Theorem 1. If $F_1(x)$ and $F_2(y)$ are known, then any distribution function $H(x, y)$ is completely determined by the sequence of characteristic coefficients (1).

Theorem 2. If the condition

$$\sum_{k,s=1}^{\infty} \frac{|\lambda_{F_1F_2}(k,s,H)|}{ks} < \infty, \tag{10}$$

is satisfied, then $H(x, y)$ is continuous, and the series (6′) and (6″) converge absolutely and uniformly over the entire plane.

An analogous condition for the continuity of $H(x, y)$ is formulated in the general case, when $H(x, y)$ is represented by the series (6).

By virtue of the orthogonality of trigonometric functions of multiple arcs, the distance (9) obtains the following expressions respectively in the general case, under condition (8), and under conditions (8) and (8′):

$$\rho^2(H; F_1F_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\Delta(x,y) + \lambda_{00}]^2 \, dF_1(x)\, dF_2(y) + \\ + \frac{1}{4\pi^4} \sum_{k,s=1}^{\infty} \frac{a_{ks}^2 + 3b_{ks}^2 + 3c_{ks}^2 + 5d_{ks}^2}{k^2s^2}, \tag{11}$$

$$\rho^2(H, F_1F_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Delta^2(x,y)\, dF_1(x)\, dF_2(y) + \frac{1}{4\pi^4} \sum_{k,s=1}^{\infty} \frac{\left|\lambda_{F_1F_2}(k,s,H)\right|^2}{k^2s^2}, \tag{11'}$$

$$\rho^2(H; F_1F_2) = \frac{1}{4\pi^4} \sum_{k,s=1}^{\infty} \frac{\left|\lambda_{F_1F_2}(k,s,H)\right|^2}{k^2s^2}. \tag{11''}$$

Let there be a sequence of bivariate distribution functions $\{H_{nm}(x, y)\}$; by $\{H_n^{(1)}(x)\}$, $\{H_m^{(2)}(y)\}$ we denote the sequences of the corresponding marginal distributions, and by the c.c. of the functions of this sequence we denote

$$\lambda_{F_1F_2}(k, s, H_{nm}) = \tilde{a}_{ks} - \tilde{d}_{ks} + i(\tilde{b}_{ks} + \tilde{c}_{ks}). \tag{2'}$$

Then, for the square of the distance between $H(x, y)$ and $H_{nm}(x, y)$, we obtain the expression

$$\rho^2(H;H_{nm}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [H_{nm}(x,y) - H(x,y)]^2\, dF_1(x)\, dF_2(y) =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{[H_n^{(1)}(x) - H_1(x)]F_2(y) + [H_m^{(2)}(y) - H_2(y)]F_1(x) + \lambda_{00} - \tilde{\lambda}_{00}\}^2\, dF_1(x)\, dF_2(y)$$

$$+\frac{1}{4\pi^4}\sum_{k,s=1}^{\infty}\frac{1}{k^2s^2}\Big[(a_{ks}-\tilde{a}_{ks})^2 + 3(b_{ks}-\tilde{b}_{ks})^2$$

$$+3(c_{ks}-\tilde{c}_{ks})^2 + 5(d_{ks}-\tilde{d}_{ks})^2\Big], \qquad (12)$$

where $\tilde{\lambda}_{00}$ for $H_{nm}(x,y)$ is defined analogously to λ_{00} (see (7)). From (12) it follows directly that

Theorem 3. *For the convergence in the mean of the sequence $\{H_{nm}(x,y)\}$ to the limiting distribution function $H(x,y)$ as $n,m \to \infty$, it is necessary and sufficient that all terms of the convergent series (12) tend to zero.*

3. Suppose that the distribution functions $H(x,y)$, $F_1(x)$, and $F_2(y)$ have densities $h(x,y)$, $p_1(x)$, $p_2(y)$, respectively, with $p_1(x) > 0$, $p_2(y) > 0$. Then at points of continuity of $h(x,y)$ we obtain the expansion

$$\frac{h(x,y)}{p_1(x)p_2(y)} = 1 + 2\sum_{k=1}^{\infty}\{\alpha_{F_1}(k,H_1)\cos kz_1 + \beta_{F_1}(k,H_1)\sin kz_1\}$$

$$+2\sum_{s=1}^{\infty}\{\alpha_{F_2}(s,H_2)\cos sz_2 + \beta_{F_2}(s,H_2)\sin sz_2\} + 4\sum_{k,s=1}^{\infty}[a_{ks}\cos kz_1\cos sz_2 +$$

$$+b_{ks}\sin kz_1\cos sz_2 + c_{ks}\cos kz_1\sin sz_2 + d_{ks}\sin kz_1\sin sz_2]. \qquad (13)$$

If

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{h^2(x,y)}{p_1(x)p_2(y)}\, dx\, dy < \infty, \qquad (14)$$

then the nonnegative quantity

$$\rho^2(h;p_1p_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{h^2(x,y)}{p_1(x)p_2(y)}\, dx\, dy - 1 = 2\sum_{k=1}^{\infty}\left|\lambda_{F_1}(k,H_1)\right|^2 +$$

$$+2\sum_{s=1}^{\infty}\left|\lambda_{F_2}(s,H_2)\right|^2+4\sum_{k,s=1}^{\infty}\left(a_{ks}^2+b_{ks}^2+c_{ks}^2+d_{ks}^2\right) \tag{15}$$

will be called the square of the distance from the density $h(x,y)$ to the density $p_1(x)p_2(y)$.

If, for the sequence of densities $\{h_{nm}(x,y)\}$, the integrals of the form (14) are finite, then for the distance between the densities $h_{nm}(x,y)$ and $h(x,y)$ we obtain the expression

$$\rho^2(h_{nm};h)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[h_{nm}(x,y)-h(x,y)]^2}{p_1(x)p_2(y)}\,dx\,dy=$$

$$=2\sum_{k=1}^{\infty}\left|\lambda_{F_1}(k,H_n^{(1)})-\lambda_{F_1}(k,H_1)\right|^2+2\sum_{s=1}^{\infty}\left|\lambda_{F_2}(s,H_m^{(2)})-\lambda_{F_2}(s,H_2)\right|^2+$$

$$+4\sum_{k,s=1}^{\infty}\left[(a_{ks}-\tilde{a}_{ks})^2+(b_{ks}-\tilde{b}_{ks})^2+(c_{ks}-\tilde{c}_{ks})^2+(d_{ks}-\tilde{d}_{ks})^2\right]. \tag{16}$$

From the last formula it follows immediately that

Theorem 4. *For convergence in the mean of the sequence of densities $\{h_{nm}(x,y)\}$ to the limiting density $h(x,y)$, it is necessary and sufficient that the series (16) converge and that its sum tend to zero as $n,m\to\infty$.*

Remark. Let the one-dimensional densities $H(x,y)$ coincide with $p_1(x)$ and $p_2(y)$; then formula (15) takes the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{h^2(x,y)}{p_1(x)p_2(y)}\,dx\,dy=1+4\sum_{k,s=1}^{\infty}\left(a_{ks}^2+b_{ks}^2+c_{ks}^2+d_{ks}^2\right). \tag{15'}$$

In this case the bilinear expansion of the density is valid

$$h(x,y)=p_1(x)p_2(y)\left[1+\sum_{k=1}^{\infty}\lambda_k\varphi_k(x)\psi_k(y)\right] \tag{17}$$

in the eigenfunctions of the kernel $h(x,y)/\sqrt{p_1(x)p_2(y)}$, and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{h^2(x,y)}{p_1(x)p_2(y)}\,dx\,dy=1+\sum_{k=1}^{\infty}\lambda_k^2. \tag{18}$$

From (15) and (18) follows the equality

$$\sum_{k=1}^{\infty} \lambda_k^2 = 4 \sum_{k,s=1}^{\infty} (a_{ks}^2 + b_{ks}^2 + c_{ks}^2 + d_{ks}^2)\,, \tag{19}$$

which relates the characteristic coefficients (3) to the eigenvalues λ_k, and expression (19) is an exhaustive measure of the dependence ([2]) between the random variables x and y.

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Note: Figure translations are in progress. See original paper for figures.

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