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Abstract

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MATHEMATICAL PHYSICS

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SPACES OF TEST AND GENERALIZED FUNCTIONS OF EXPONENTIAL TYPE

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1. Since the introduction by Dirac of the concept of the δ-function, test and generalized functions have played a fundamental role in theoretical physics. For example, in constructing the mathematical foundations of quantum mechanics (^{1}) (the spectral theory of linear operators) it was found that the framework of Hilbert space is too narrow for the needs of this theory; extensions of Hilbert spaces to spaces of generalized functions (^{2}) or rigged Hilbert spaces (^{3}) are necessary. As is well known, the spectral theory of operators of the form p^k (p is a differential operator, k is a number) is well served by a certain type of spaces of test and generalized functions introduced by L. Schwartz (^{2}). We shall call the original Schwartz spaces spaces of test and generalized functions of power type.

In studying infinite-dimensional representations of compact Lie groups, in particular the rotation group R_3 (^{4}), we shall need a new type of spaces of test and generalized functions, which we shall call **spaces of exponential type**. This type of space is well adapted to the study of operators of the form K^p (K is any complex number). The present note is devoted to a description of this type of space.

2. Let Φ be a linear space. In the abstract model the elements φ of the space Φ are infinite numerical sequences $\varphi = (\varphi_0, \varphi_1, \varphi_2, ...)$. Consider the scalar product

$$(\varphi, \psi)_K = \sum_{n=0}^{\infty} |K|^{2n} \varphi_n \overline{\psi}_n, \tag{1}$$

where K is any complex number. We shall consider in Φ only a countable system of scalar products of the form (1): $|K| = 1, 2, ...$ (so that the first axiom of countability is satisfied). Obviously, for any $\varphi \in \Phi$ the inequalities

$$(\varphi, \varphi)_1 \leq (\varphi, \varphi)_2 \leq \cdots \leq (\varphi, \varphi)_K \leq \cdots \tag{2}$$

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hold.

It can be shown that the norms generated by the scalar products (2) are pairwise compatible (see the examples). Then the completions H_K of the space Φ with respect to the scalar products $(\varphi, \psi)_K$ are connected by the inclusion (H_K, $|K| = 1, 2, \ldots$ are complete Hilbert spaces)

$$H_1 \supset H_2 \supset \cdots \supset H_K \supset \cdots \supset \Phi.$$

The complete countably Hilbert space Φ is the intersection of all H_K:

$$\Phi = \bigcap_{|K|=1}^{\infty} H_K.$$

The topology in Φ is introduced as follows. A complete countable system of neighborhoods of zero of the space Φ is given by the sets $U_{K,1/m}(0)$,

$|K| = 1, 2, \ldots;\ m = 1, 2, \ldots,$ consisting of all $\varphi \in \Phi$ for which $(\varphi, \varphi)_p < 1/m,\ |p| = 1, 2, \ldots, |K|$.

A sequence $\varphi_N \in \Phi$ converges to $\varphi \in \Phi$ in the (original) topology in Φ if $(\varphi_N - \varphi, \varphi_N - \varphi)_K \to 0$ for every K as $N \to \infty$.

Definition 1. A countably Hilbert space in which the topology is given by a countable set of (compatible) scalar products of the form (1) will be called a **space of test functions of exponential type**.*

It is known that every complete countably normed space can be made into a complete Fréchet metric space (^{5}) if one defines in it the distance

$$\rho(\varphi, \psi) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|\varphi - \psi\|_k}{1 + \|\varphi - \psi\|_k}.$$

In our case $\|\varphi\|_K = \sqrt{(\varphi, \varphi)_K}$.

We now consider the space Φ', conjugate to Φ with respect to the scalar product $(f, \varphi)_1,\ f \in \Phi',\ \varphi \in \Phi$. It is the union of an increasing sequence of complete Hilbert spaces H'_K, conjugate to H_K:

$$H'_1 \subset H'_2 \subset \ldots \subset H'_K \subset \ldots \subset \Phi', \quad \text{i.e.} \quad \Phi' = \bigcup_{|K|=1}^{\infty} H'_K;$$

Φ' is the space of linear functionals on Φ.

Definition 2. The space Φ', conjugate to the space of test functions of exponential type Φ, will be called the **space of generalized functions of exponential type**.

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In Φ' we shall consider only the weak topology. A weak neighborhood of zero in Φ' is specified by a number $\varepsilon > 0$ and by an arbitrary finite set $\varphi_1, \varphi_2, \dots, \varphi_n \in \Phi$ as the totality of all $f \in \Phi'$ for which $|(f, \varphi_p)_1| < \varepsilon,\ p = 1, 2, \dots, n$. A sequence $f_N \in \Phi'$ converges weakly to f if $(f_N, \varphi)_1 \to (f, \varphi)_1$ for all $\varphi \in \Phi$.

The space Φ' can be made into a metric space (^{6}) if one defines in it the distance

$$\rho(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(f-g)}{1 + p_k(f-g)}, \qquad f, g \in \Phi',$$

where $p_k(f) = |(f, \varphi_k)_1|,\ k = 1, 2, \ldots$, is a countable number of seminorms defining the weak topology.

Next, in each of the H_K consider the orthonormal systems

$$^{(K)}e_n = \frac{1}{|K|^n}\,{}^{(1)}e_n, \qquad ({}^{(K)}e_n, {}^{(K)}e_{n'})_K = \delta_{nn'}.$$

$\{{}^{(K)}e_n\}_{n=0}^{\infty}$ is a complete system in H_K, so that every element $\varphi \in H_K$ can be represented in the form

$$\varphi = \sum_{n=0}^{\infty} (\varphi, {}^{(K)}e_n)_K\, {}^{(K)}e_n.$$

For $|K| < |K'|$ the mapping $T_{K'}^{K}$ of the space $\tilde{H}_K$ into $\tilde{H}_{K'}$ has the form

$$T_{K'}^{K}\, {}^{(K)}\varphi = \sum_{n=0}^{\infty} \lambda_n (\varphi, {}^{(K)}e_n)_K\, {}^{(K')}e_n \in H_{K'}, \qquad {}^{(K)}\varphi \in H_K,$$

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* In the abstract model, Schwartz' original spaces are specified by a countable system of scalar products of the form

$$(\varphi, \varphi)_k = \sum_{n=0}^{\infty} n^{2k} |\varphi_n|^2, \qquad k = 0, 1, 2, \dots.$$

The synthesis of these scalar products and the scalar products of the form (1) is

$$(\varphi, \varphi)_{K,p} = \sum_{n=0}^{\infty} |K|^{2n} n^{2p} |\varphi_n|^2, \qquad |K| = 1, 2, \dots;\ p = 0, 1, 2, \dots.$$

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where $\lambda_n = \left|\frac{K}{K'}\right|^n$. Since the series $\sum_{n=0}^{\infty}\lambda_n = \frac{|K'|}{|K'|-|K|}$ converges, the mapping $T_{K'}^{K}$ is nuclear (³). Thus, the following holds:

Theorem. *Countably Hilbert spaces of exponential type are nuclear.*

This fact means that all bounded sets in Φ are compact (such spaces are called perfect); weak convergence in Φ coincides with strong convergence (the latter, as is known (⁵), in the case of countably normed spaces coincides with the original one); a countable sum of bounded sets in Φ does not give the whole of Φ (bounded sets in Φ are nowhere dense).

3. Examples.

A. Consider the Hilbert space H^+ of square-integrable functions $f(x)$ on the interval $[0, 2\pi]$:
$(f, f) = \int_0^{2\pi} |f(x)|^2 dx$. The system of functions $\{e^{inx}\}_{n=0}^{\infty}$ forms a complete system (an orthonormal basis) in H^+, so that any function $f(x) \in H^+$ can be represented in the form

$$f(x) = \sum_{n=0}^{\infty} f_n e^{inx}, \qquad (f, f) = \sum_{n=0}^{\infty} |f_n|^2.$$

In this realization, the space of test functions of exponential type Φ is formed by functions

$$\varphi(x) = \sum_{n=0}^{\infty} \varphi_n e^{inx},$$

square-integrable on the interval $[0, 2\pi]$ under any of its parallel translations in the complex plane $(x)^*$, i.e., if $\varphi \in \Phi$, then also $L_K\varphi(x) = \varphi(x - i\ln K) \in \Phi$, where $0 \leq |K| < \infty$, and

$$(\varphi, \varphi) = \sum_{n=0}^{\infty} |\varphi_n|^2, \qquad (L_K\varphi, L_K\varphi) = \sum_{n=0}^{\infty} |K|^{2n}|\varphi_n|^2 = (\varphi, \varphi)_K.$$

(An example of such a function is $\varphi(x) = \exp(e^{ix})$.)

If we denote $e^{ix} = z$, then Φ is formed by all entire analytic functions $\varphi(z)$ of the complex variable z (these functions are square-integrable on the circle of any radius $|z| = |K|$). It follows from this that the scalar products (1), (2) are pairwise consistent. Indeed, if a sequence of entire analytic functions $\varphi_N(z)$ converges to $\varphi(z)$, this means that for any K, $\|\varphi_N - \varphi\|_K \to 0$ as $N \to 0$. Thus, if $|K| > |K'|$ and $\|\varphi_N\|_{K'} \to 0$, then also $\|\varphi_N\|_K \to 0$, i.e., the scalar products (2) are consistent.

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In the realization under consideration, the space of generalized functions Φ' (linear functionals on Φ), in addition to entire functions, also includes analytic functions $f(z)$ with singularities concentrated in any part of the complex plane $z = e^{ix}$. Here the elements $f \in \Phi'$ are germs of analytic functions $f(z)$, considered only in their Mittag-Leffler star (7), where they are represented by series

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

with coefficients $|f_n| \to |K|^n$ as $n \to \infty$, $0 \leq |K| < \infty$. Series of this type we shall call **summable in the sense of convergen-**

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* In the realization under consideration, the original Schwartz space of test functions (2) is formed by functions $\varphi(x)$ square-integrable together with all their derivatives on $[0, 2\pi]$. An example of such a function may be

$$\varphi(x) = (1 - ae^{ix})^{-1}, \qquad |a| < 1.$$

in Φ'.* An example may be the function $f(z) = (1 - Az)^{-1}$, $0 \leq |A| < \infty$.

B. Another example of a space $\Phi' \supset H \supset \Phi$ is the class of entire functions of the complex variable $z = x + iy$, considered in (7,8). In this realization the Hilbert space H is formed by entire analytic functions of order $\rho \leq 2$ and type τ ($0 \leq \tau < \infty$, if $\rho < 2$, and $0 \leq \tau < 1/2$, if $\rho = 2$).

The scalar product in H is the form

$$(f, g) = \int f(z)\overline{g}(z)\, d\mu(z), \quad \text{where } d\mu(z) = \frac{1}{\pi} e^{-|z|^2}\, dz, \quad dz = dx\, dy.$$

The space of basic functions of exponential type Φ is formed by entire analytic functions of order $\rho < 2$ and type $0 \leq \tau < \infty$, while the conjugate space Φ' is formed by entire analytic functions of order $\rho \leq 2$ and type $0 \leq \tau < \infty$.

4. In conclusion let us make several remarks concerning infinite-dimensional irreducible representations of compact Lie groups. The basis for introducing the topology considered above in the representation space Φ_λ (λ is a set of complex numbers specifying an irreducible representation $D(\lambda)$ of the Lie group G) is the existence of a bilinear functional (4) (cf. (10))

$$[f^{(\lambda)}, \varphi^{(\bar{\lambda})}]_\lambda = (f^{(\lambda)}, I_{\bar{\lambda}} \varphi^{(\bar{\lambda})})_\lambda \tag{3}$$

with the following properties: a) $f^{(\lambda)} \in \Phi'_\lambda$, $\varphi^{(\bar{\lambda})} \in \Phi_{\bar{\lambda}}$, where $\Phi_{\bar{\lambda}}$ and Φ'_λ are spaces of basic and generalized functions of the type considered ($\Phi_{\bar{\lambda}}$ is a countably normed space; Φ'_λ is the space of linear continuous

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functionals on Φ_λ; with respect to (3) the representations $D(\lambda)$ and $D(\bar{\lambda})$ are conjugate to one another); b) (3) is a nondegenerate functional, and

$$[f^{(\lambda)}, \varphi^{(\bar{\lambda})}]_\lambda = [f^{(\bar{\lambda})}, \varphi^{(\lambda)}]_{\bar{\lambda}};$$

c)

$$[f^{(\lambda)}, T_{\bar{\lambda}}^{-1}(g)\varphi^{(\bar{\lambda})}]_\lambda = [T_\lambda(g) f^{(\lambda)}, \varphi^{(\bar{\lambda})}]_\lambda,$$

although $T_{\bar{\lambda}}^{-1}(g)\varphi^{(\bar{\lambda})} \notin \Phi_{\bar{\lambda}}$, where $g \to T_\lambda(g)$, $g \in G$, is the representation $D(\lambda)$ of the group G in Φ'_λ.

For real λ, (3) turns into a strongly indefinite scalar product.**

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* The method of summation presented is a functional analogue of the classical summation methods of Le Roy and Mittag-Leffler ([7]).

** The remarks given concern only the so-called discrete series of representations. We do not touch upon the continuous series of representations here.

Note: Figure translations are in progress. See original paper for figures.

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