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ON A POSTERIORI ERROR ESTIMATES

MATHEMATICS

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Abstract

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MATHEMATICS

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ON A POSTERIORI ERROR ESTIMATES

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1. Consider the equation

$$Gx = 0 \tag{1}$$

with a sufficiently smooth nonlinear operator G acting from one Banach space E into another space F. Suppose that, by some means, a point $x_0 \in E$ has been chosen, which we regard as an approximate solution of equation (1). The point x_0 may have been obtained as the result of applying some approximate method, found from physical considerations, selected as the result of a random search, etc.—for the subsequent reasoning this plays no role. The "quality" of the point x_0 as an approximate solution of equation (1) can be characterized by the magnitude (norm) of the residual Gx_0. Often, however, error estimates are needed.

Denote by $\mathfrak{M}$ the set of all solutions of equation (1). The error of the approximate solution x_0 will mean the distance $\delta(x_0)$ from the point x_0 to the set $\mathfrak{M}$; if $\mathfrak{M}$ is empty, we set $\delta(x_0) = \infty$. Obtaining an estimate $\delta(x_0) \le \delta_0$ is equivalent to proving the existence of at least one solution of equation (1) in the ball $T = \{x : \|x - x_0\| \le \delta_0\}$. This principle transforms every method of proving existence theorems (see, for example, ($^{1-3}$)) into a method for obtaining a posteriori error estimates.

Let us first consider equation (1) of the special form

$$x = Ax \tag{2}$$

with a completely continuous operator A. Suppose the operator $A'(x_0)$ exists, and

$$\|Ax - Ax_0 - A'(x_0)(x - x_0)\| \le a(\|x - x_0\|) \qquad (\|x - x_0\| \le r_0),$$

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where $a(r) = o(r)$ (for example, $a(r) = ar^2$). Let the operator $\Gamma = [I - A'(x_0)]^{-1}$ be defined. Then the Schauder principle implies the obvious estimate $\delta(x_0) \leq \delta_0$, if $\delta_0 \leq r_0$ and

$$\|\Gamma(Ax_0 - x_0)\| \leq \delta_0 - \|\Gamma\| a(\delta_0). \tag{3}$$

Usually the second term on the right-hand side of inequality (3) is neglected, and the approximate error estimate

$$\delta(x_0) \lesssim \|\Gamma(Ax_0 - x_0)\| \tag{4}$$

is used.

Return to equation (1). Suppose that

$$Gx = Gx_0 + G'(x_0)(x - x_0) + \omega(x - x_0),$$

where $G'(x_0)$ has a continuous inverse Γ, $\omega(h) \leq a(\|h\|)$ for $\|h\| \leq r_0$, and $\|\omega(x) - \omega(y)\| \leq q(r)\|x - y\|$ for $\|x\|, \|y\| \leq r \leq r_0$. If $\delta_0 \leq r_0$ and

$$\|\Gamma G x_0\| \leq \delta_0 - \|\Gamma\| a(\delta_0), \qquad \|\Gamma\| q(\delta_0) < 1, \tag{5}$$

then from the contraction mapping principle there follows the estimate $\delta(x_0) \leq \delta_0$. Usually $q(r) \to 0$ as $r \to 0$, and therefore the second condition (5) is satisfied for small δ_0; the value δ_0 is then determined from the first condition (5), neglecting the second term on the right-hand side. As a result one obtains the estimate analogous to (4),

$$\delta(x_0) \leq \|\Gamma G x_0\|. \tag{4}$$

The estimates given are well known. However, they lose their meaning when the operator Γ is undefined or when the norm of the operator Γ is large. Such situations are always encountered if multiple or "almost multiple" (in one sense or another) solutions of the equation are sought. In these cases more delicate considerations must be applied.

2. Let us again consider equation (2) with a completely continuous operator A. Suppose that x_0 is an isolated fixed point of the operator $Bx = Ax - Ax_0 + x_0$. Let $\gamma(x_0; B)$ be the index (see ([2],[4])) of this fixed point. The second (after the index) important characteristic of the fixed point x_0 is the function

$$\alpha_0(r) = \inf_{\|x - x_0\| = r} \|Bx - x\|_* \quad (r \leq r_0), \tag{6}$$

where $\|\cdot\|_*$ is some norm in E. Since it is impossible to find $\alpha_0(r)$ explicitly, in applications of the theorem formulated below one may use its minorants, i.e., such functions $\alpha(r)$ that $\alpha_0(r) \geq \alpha(r)$ for $r \leq r_0$.

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Theorem 1. Let $\gamma(x_0; B) \neq 0$. Suppose that the residual $Ax_0 - x_0$ satisfies, for some $\delta_0 \in [0, r_0]$, the inequality $\|Ax_0 - x_0\|_* \leq \alpha_0(\delta_0)$. Then $\delta(x_0) \leq \delta_0$.

A general algorithm for computing the index $\gamma(x_0; B)$ for cases when $I - A'(x_0)$ is noninvertible was developed in ([5],[6]). We shall use its special variant, indicated already in ([2]). Let us also note the papers ([7],[8]). We indicate one concrete realization of the evident Theorem 1.

Suppose that the completely continuous operator A admits the representation

$$Ax = Ax_0 + A'(x_0)(x - x_0) + A_k(x - x_0) + T(x - x_0), \tag{7}$$

where A_k is a homogeneous form of order k; T is an operator containing terms of order of smallness higher than k.

Let 1 be an eigenvalue of the linear operator $A'(x_0)$, to which there corresponds the root subspace E_0, consisting only of eigenvectors; denote by E^0 a subspace complementary to E_0 and invariant for $A'(x_0)$. Each point $x \in E$ is uniquely representable in the form $x = u + v$, where $u \in E_0$ and $v \in E^0$. The equalities $P_0 x = u$, $P^0 x = v$ define the projectors onto E_0 and E^0. Denote the restriction of the operator $A'(x_0)$ to E^0 by A_1^0; 1 is not an eigenvalue of the operator A_1^0; therefore there exists an operator defined on E^0 and with values in E^0,

$$\Gamma^0 = (I - A_1^0)^{-1}.$$

Introduce for consideration the vector field $\Phi_0 u = -P_0 A_{ku}$ on the finite-dimensional space E_0. If an affine coordinate system is introduced in E_0, then the components of the field Φ_0 will be homogeneous polynomials of degree k in the coordinates. We shall assume that

$$\|P_0 A_{ku}\| \geq b_1 \|u\|^k \quad (u \in E_0); \tag{8}$$

then the field Φ_0 vanishes only at the point θ; denote the index of this zero of the field Φ_0 by γ_0. Introduce also constants q_1, b_2 and b_3 such that

$$\|P_0 A_{kh} - P_0 A_{kP} 0h\| \leq q_1 \|h\|^{k-1} \|P^0 h\|,$$

$$\|P^0 A_{kh}\| \leq b_2 \|h\|^k, \qquad \|Th\| \leq b_3 \|h\|^{k+1} \quad (h \in E;\ \|h\| \leq r_0).$$

Theorem 2. Suppose that the completely continuous operator A admits representation (7), with 1 an eigenvalue of the operator $A'(x_0)$ and $\cdots$

condition (8) is fulfilled. Let $\gamma_0 \neq 0$. Let the residual $Ax_0 - x_0$ satisfy the inequality

$$\|Ax_0 - x_0\| \leq \frac{b_1 \delta_0^k - \|\Gamma^0\| b_2 (k b_1 + q_1) \delta_0^{2k-1}}{1 + \|\Gamma^0\| (k b_1 + q_1) \delta_0^{k-1}} - b_3 \delta_0^{k+1}, \tag{9}$$

where $\delta_0 \leq r_0$. Then $\delta(x_0) \leq \delta_0$.

The computation of γ_0 in the cases of one-dimensional and two-dimensional E_0 is carried out by known formulas. If $\dim E_0 \geq 3$, then the computation of γ_0 becomes a difficult problem already for $k = 2$; however, in a number of cases here too one can show that γ_0 is different from zero.

Theorem 2 mainly has to be applied when the residual is small. Then δ_0 is also small, and therefore in the right-hand side of inequality (9) one may discard terms of higher order of smallness. We then arrive at the approximate error estimate:

$$\delta(x_0) \lesssim b_1^{-1/k} \|Ax_0 - x_0\|^{1/k}. \tag{10}$$

It is apparently this estimate that should be recommended.

3. Theorem 2 loses its force if $\gamma_0 = 0$. To obtain error estimates in this case one has to take into account not only the magnitude of the residual, but also its direction. We restrict ourselves to the case of one-dimensional E_0; then $\gamma_0 = 0$ if k is even; the operator P_0 has the form $P_0 x = l(x)e$, where l is a linear functional, $e \in E_0$, $\|e\| = 1$.

Theorem 3. Let the completely continuous operator A admit the representation (7), where 1 is an eigenvalue of the operator $A'(x_0)$, the subspace E_0 is one-dimensional, and k is even. Let the residual $Ax_0 - x_0$ satisfy the conditions

$$l(Ax_0 - x_0)l(A_{ke}) < 0$$

and

$$\frac{1}{\|\Gamma^0\|}\rho_0 - b_2(\rho_0 + \delta_0)^k - \|P^0\| b_3(\rho_0 + \delta_0)^{k+1} \geq \|P^0(Ax_0 - x_0)\|,$$

$$b_1\delta_0^k - q_1(\rho_0 + \delta_0)^{k-1}\rho_0 - \|P_0\| b_3(\rho_0 + \delta_0)^{k+1} \geq$$

$$\geq \|P_0(Ax_0 - x_0)\| \geq q_1\rho_0^k + \|P_0\| b_3\rho_0^{k+1},$$

where $\rho_0 \leq \delta_0$, $\rho_0 + \delta_0 \leq r_0$. Then $\delta(x_0) \leq \rho_0 + \delta_0$.

In the case when the residual $Ax_0 - x_0$ is small and the projections $P^0(Ax_0 - x_0)$ and $P_0(Ax_0 - x_0)$ have the same order of magnitude, Theorem 3 also yields the approximate error estimate (10).

4. Let us return to equation (1) with an operator G acting from the space E into the space F. We shall assume that

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$$Gx = Gx_0 + G'(x_0)(x - x_0) + G_k(x - x_0) + T(x - x_0).$$

Suppose that $G'(x_0)$ is normally solvable and has zero Noether index (the set of zeros of the operator $G'(x_0)$ is finite-dimensional, and the set of zeros of the adjoint operator has the same dimension). The space F can be represented as the direct sum of the subspace $F^0 = G'(x_0)E$ and of some finite-dimensional subspace F_0; every element $f \in F$ is then representable in the form $f = z + w$, where $z \in F_0$ and $w \in F^0$; this representation determines the projectors $Q_0 f = z$ and $Q^0 f = w$. We denote the set of zeros of the operator $G'(x_0)$ by E_0, and some direct complement to it by E^0; let the projectors P_0 and P^0 be defined as in the preceding item. The restriction of the operator $G'(x_0)$ to E^0 has an inverse Γ^0 on F^0.

We shall assume that the homogeneous form G_k of order k and the operator T satisfy the natural estimates:

$$\|\Gamma^0 Q^0[G_k(u+v_1) - G_k(u+v_2) + T(u+v_1) - T(u+v_2)]\| \leq$$

$$\leq p_1(\rho+\delta)^{k-1}\|v_1 - v_2\|,$$

$$\|\Gamma^0 Q^0(G_k u + Tu)\| \leq c_1\|u\|^k,$$

$$\|Q_0 G_k(u+v) - Q_0 G_k u\| \leq p_2(\rho+\delta)^{k-1}\|v\|,$$

$$\|Q_0 T(u+v)\| \leq c_2(\rho+\delta)^{k+1},$$

where $u \in E_0;\ v, v_1, v_2 \in E^0;\ \|u\| \leq \delta \leq r_1;\ \|v\|, \|v_1\|, \|v_2\| \leq \rho \leq r_2$.

In addition, we shall assume that

$$\|Q_0 G_k u\| \geq c_0\|u\|^k \qquad (u \in E_0). \tag{11}$$

The operator $Q_0 G_k$ maps the finite-dimensional space E_0 into the space F_0 of the same dimension. It follows from (11) that there is defined (see, for example, ([9,10])) the degree γ_* of the mapping $Q_0 G_k$ of a neighborhood of zero $\theta \in E_0$ relative to the zero of the space F_0. This degree of the mapping is computed in the same way as the rotation of vector fields is computed.

Theorem 4. Suppose that condition (11) is satisfied and that $\gamma_* \neq 0$. Suppose that the residual Gx_0 satisfies the inequalities

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$$\|Q^0 G x_0\| \le \rho_0 - c_1 \delta_0^k - p_1 (\delta_0 + \rho_0)^{k-1} \rho_0,$$

$$\|Q_0 G x_0\| \le c_0 \delta_0^k - p_2 (\delta_0 + \rho_0)^{k-1} \rho_0 - c_2 (\delta_0 + \rho_0)^{k+1},$$

where $\delta_0 \le r_1, \ \rho_0 \le r_2$. Then $\delta(x_0) \le \delta_0 + \rho_0$.

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