MATHEMATICS

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Abstract

Full Text

MATHEMATICS

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DIFFUSION OF TENSORS

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1. Let an affine connection Γ^i_{jk} be given on a smooth l-dimensional manifold E of class C^3, and let X be a diffusion process on E with generating differential operator

$$\mathfrak{D} = a^{ij}\nabla_i\nabla_j + b^i\nabla_i. \tag{1}$$

(∇_i is the covariant differentiation corresponding to the connection Γ^i_{jk}.) The totality $R^n_m(x)$ of all tensors of valence (m, n)* attached at the point $x \in E$ forms a linear space. The union of $R^n_m(x)$ over all $x \in E$ will be denoted by R^n_m. It will be shown that the process X induces a diffusion process U in the space R^n_m. Denote by L^n_m the space of linear continuous functions on R^n_m.**

It is shown that the semigroup of linear operators T_t corresponding to the process U leaves the space L^n_m invariant. The infinitesimal operator A of the semigroup $\mathcal{T}_t$ induced in L by the semigroup T_t is computed. The result can be described as follows. Let $u \in R^m_n$ and $w \in R^n_m$. Put

$$(u, w) = u^{j_1 \cdots j_n}_{i_1 \cdots i_m} w^{i_1 \cdots i_m}_{j_1 \cdots j_n}.$$

Every function f from L^n_m can be described by the formula

$$f(u) = (u, w(x)) \quad \text{for } u \in R^n_m(x), \tag{2}$$

where $w(x)$ is some tensor field of valence (m, n). This defines a natural identification of the space L^n_m with the space $\mathcal{L}^n_m$ of continuous tensor fields on E of valence (n, m). The infinitesimal operator A acts on such fields according to the formula

$$Aw(x) = a^{ij}(x)\nabla_i\nabla_j w(x) + b^i(x)\nabla_i w(x), \tag{3}$$

i.e. it formally has the same form as the operator $\mathfrak{D}$ (see (1)).

2. Put

$$\Gamma_\alpha u^{j_1\cdots j_n}_{i_1\cdots i_m} = \sum_\mu \Gamma^\beta_{\alpha j_\mu} u^{j_1\cdots j_n}_{i_1\cdots i_{\mu+1}\beta i_{\mu+1}\cdots i_m} - \sum_\nu \Gamma^{i_\nu}_{\alpha\beta} u^{j_1\cdots j_{\nu-1}\beta j_{\nu+1}\cdots j_n}_{i_1\cdots i_n}. \tag{4}$$

It is easy to verify that

$$(\Gamma_\alpha u, w) = -(u, \Gamma_\alpha w) \quad (u \in R^n_m(x),\ w \in R^m_n(x)). \tag{5}$$

Let $y(s)$ $(s_0 \le s \le s_1)$ be a smooth curve. A family of tensors $u(s) \in R^n_m(y_s)$ is called parallel if

$$\dot u(s) = (\Gamma_\alpha u(s))\dot y^\alpha(s). \tag{6}$$

Integrating the differential equation (6) with the initial condition $u(s_0) = h$, we define the parallel transport of a vector $h \in R^n_m(y(s_0))$ along the curve $y(s)$. Omitting the argument s, we shall write the equation—

* By a tensor of valence (m, n) we mean a tensor of the type

$$u^{j_1\cdots j_n}_{i_1\cdots i_m},$$

i.e. a tensor m times co- and n times contravariant.

** A function defined on R^n_m is called linear if it is linear on each space $R^n_m(x)$ $(x \in E)$.

···equation (6) in abbreviated form

$$\dot u = (\Gamma_\alpha u)\dot y^\alpha. \tag{7}$$

3. The diffusion process X corresponding to the generating operator $\mathfrak{D}$ can be constructed as the solution of the system of stochastic differential equations of Ito

$$dx^\alpha(t) = \rho^\alpha[x(t)]\,dt + \sigma^\alpha_\beta[x(t)]\,d\xi^\beta(t), \tag{8}$$

where $\xi^1(t), \ldots, \xi^l(t)$ are l independent Wiener processes. It is known (see, for example, ([1]), Sec. 11.12) that the generating differential operator of the process $x(t)$ is given by the formula

$$\mathfrak{D} f = \rho^\alpha \partial_\alpha f + \tfrac{1}{2}\tilde a^{\alpha\beta}\partial_\alpha\partial_\beta f, \tag{9}$$

where

$$\tilde{a}^{\alpha\beta} = \sum_{\gamma=1}^{l} \sigma_\gamma^\alpha \sigma_\gamma^\beta. \tag{10}$$

The covariant derivative ∇_α is related to the partial derivative ∂_α by the relation

$$\nabla_\alpha = \partial_\alpha - \Gamma_\alpha. \tag{11}$$

Therefore, for

$$\rho^\alpha = b^\alpha - a^{\beta\gamma}\Gamma_{\beta\gamma}^\alpha, \qquad \tilde{a}^{\alpha\beta} = 2a^{\alpha\beta} \tag{12}$$

the operator defined by formula (9) coincides with the operator (1).

4. We now define stochastic parallel transport of tensors along the trajectory $x(t)$ of the diffusion process X. Consider on this trajectory two nearby points $x(t)$ and $x(t+\Delta t)$ and join them by a smooth curve $y(s)$. According to Sec. 2, a family of tensors $u(s)$, parallel along the curve $y(s)$, satisfies equation (7). We shall agree to write $Q_1 \approx Q_2$ if $Q_1 - Q_2 = o(\Delta s^2)$. We have

$$\Delta u = u(s+\Delta s) - u(s) \approx \dot{u}\Delta s + \tfrac{1}{2}\ddot{u}\Delta s^2. \tag{13}$$

On the other hand,

$$\Delta x^i = x^i(t+\Delta t) - x^i(t) = y^i(s+\Delta s) - y^i(s) \approx \dot{y}^i \Delta s + \tfrac{1}{2}\ddot{y}^i \Delta s^2.$$

Hence

$$\dot{y}^\alpha \dot{y}^\beta \Delta s^2 \approx \Delta x^\alpha \Delta x^\beta, \qquad \tfrac{1}{2}\ddot{y}^i \Delta s^2 \approx \Delta x^i - \dot{y}^i \Delta s.$$

Denote by $\Pi_{\alpha\beta}$ the operator obtained if, on the right-hand side of (4), $\Gamma_{\alpha j}^{\,i}$ is replaced by $\partial_\beta \Gamma_{\alpha j}^{\,i}$; in other words, put

$$\Pi_{\alpha\beta} u = (\partial_\beta \Gamma_\alpha) u.$$

Differentiating equation (7) with respect to s and substituting the values of $\dot{u}$ and $\ddot{u}$ into (13), we obtain

$$\Delta u \approx (\Pi_{\alpha\beta} u + \Gamma_\alpha \Gamma_\beta u)\Delta x^\alpha \Delta x^\beta + \Gamma_\alpha u \Delta x^\alpha. \tag{14}$$

The right-hand side does not depend on the choice of the smooth curve joining $x(t)$ and $x(t+\Delta t)$. This choice can affect only the value of Δs and, consequently, the meaning of the relation $\approx$. We restrict the choice of curves $y(s)$ only by the requirement that $\sum_i (\Delta x^i)^2/\Delta s^2$ be bounded above and below by positive constants as $\Delta t \to 0$. It is easy to verify that, if this requirement is satisfied in some local coordinate system at the point $x(t)$, then it is also satisfied in any coordinate system smoothly related to the original one. We may therefore take as $y(s)$ curves defined by linear equations in some admissible coordinate system (one could also have taken a segment of a geodesic). Now consider a subdivision $0 = t_0 < t_1 < \ldots < t_n = t$ of the interval $[0, t]$ and write equation (14) for each interval $[t_i, t_{i+1}]$. Passing to the limit as $\max|t_i - t_{i-1}| \to 0$

and, taking into account that $x(t)$ is given by equation (8), we arrive at the following stochastic differential equation defining stochastic parallel transport:

$$du = (\Pi_{\alpha\beta} u + \Gamma_\alpha \Gamma_\beta u)\, dx^\alpha dx^\beta + (\Gamma_\alpha u)\, dx^\alpha. \tag{15}$$

Here dx^α is expressed by formula (8), while $dx^\alpha dx^\beta$, by definition, is equal to $\tilde{a}^{\alpha\beta} dt$. Substituting these values in (15), we arrive at the equation

$$du = \{\Pi_{\alpha\beta} u + \Gamma_\alpha \Gamma_\beta u\} \tilde{a}^{\alpha\beta} + \varrho^\alpha \Gamma_\alpha u] dt + (\Gamma_\alpha u) \sigma^\alpha_\gamma d\xi^\gamma. \tag{16}$$

The system of stochastic differential equations (8) and (16) defines a diffusion process U in R^n_m. The rank of the diffusion matrix of this process does not exceed the rank of $a^{\alpha\beta}$. Therefore the process U is strongly degenerate, even if the process X is nondegenerate.

5. Let $h \in L^n_m(x_0)$. Denote by $x_h(t), u_h(t)$ the solution of the system of equations (8), (16) under the initial condition $x_h(0) = x_0,\ u_h(0) = h$. Since $x_h(t)$ is the point of application of the tensor $u_h(t)$, $x_h(t)$ is uniquely recovered from $u_h(t)$.

The semigroup T_t of the process U is defined by the formula

$$T_t f(h) = \mathbf{M} f[u_h(t)]. \tag{17}$$

It is easy to see that $u_h(t)$ is linear in h. Therefore the space L^n_m is invariant with respect to T_t. Identifying, by means of formula (2), the space L^n_m with the space $\mathcal{L}^m_n$, we note that the semigroup T_t induces in $\mathcal{L}^m_n$ a semigroup $\mathcal{T}_t$, acting according to the formula

$$(h, \mathcal{T}_t w(x)) = \mathbf{M}\, (u(t), w[x(t)]). \tag{18}$$

According to Itô' s calculus of stochastic differentials (see [2] or [1], Ch. 7),

$$d(u, w(x)) = (u, dw(x)) + (du, w(x)) + (du, dw(x)), \tag{19}$$

where du is defined by formula (15) or (16), and

$$dw^i(x) = \partial_\alpha w^i(x)\,dx^\alpha + \tfrac{1}{2}\partial_\alpha\partial_\beta w^i(x)\,dx^\alpha dx^\beta. \tag{20}$$

Expressing $\partial_\alpha w$ and $\partial_\alpha\partial_\beta w$ with the aid of formula (11) and relying on (5), we obtain

$$(u, \partial_\alpha w) = (u, \nabla_\alpha w) - (\Gamma_\alpha u, w),$$

$$\begin{aligned}(u, \partial_\alpha\partial_\beta w) = (u, \nabla_\alpha\nabla_\beta w) - (\Gamma_\beta\Gamma_\alpha u, w) - (\Pi_{\beta\alpha} u, w)\\ - (\Gamma_\beta u, \partial_\alpha w) - (\Gamma_\alpha u, \partial_\beta w) + \Gamma^\gamma_{\alpha\beta}(u, \nabla_\gamma w).\end{aligned} \tag{21}$$

From (19), (20), and (21),

$$d(u, w(x)) = (u, \nabla_\alpha w)dx^\alpha - \tfrac{1}{2}(u, \nabla_\alpha\nabla_\beta w + \xi\Gamma^\gamma_{\alpha\beta}\nabla_\gamma w)dx^\alpha dx^\beta.$$

Substituting here the value of dx^α from (8) and taking (12) into account, we have

$$d(u, w(x)) = (u, \mathfrak{D}w(x))dt + (u, \nabla_\alpha w(x))\sigma^\alpha_\varrho d\xi^\varrho.$$

Integrating from 0 to t, taking the mathematical expectation and using (18), we have

$$(h, \mathcal{T}_t w(x)) - (h, w(x)) = \int_0^t \mathbf{M}\left(u(s), \mathfrak{D}w[x(s)]\right) ds.$$

Hence it follows easily that, for tensor fields of class C^2, $Aw = \mathfrak{D}w$, i.e. formula (3) is valid.

The general theory of semigroups (see, for example, (1), Chap. 1) makes it possible to conclude that, for $w(x) \in C^2$, the tensor field $w_t = \mathcal{T}_t w$ satisfies the differential equation

$$\partial w_t / \partial t = \mathfrak{D} w_t$$

with the initial condition $w_0 = w$.

6. What has been presented is a further development of the investigations begun by K. Ito in (3). Ito considered a more special case, in which: a) in formula (1) $b^i = 0$, and a^{ij} admits an inverse matrix a_{ij}; b) the connection Γ^i_{jk} is the Riemannian connection (without torsion) corresponding to the metric tensor a_{ij}; c) the tensors u have valence $(0, n)$. Ito' s definition of stochastic parallel displacement also differs somewhat.

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REFERENCES

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[2] K. Ito, *Nagoya Math. J.*, **3**, 55 (1951); Collection of translations *Matematika*, **3**, 5, 131 (1959).
[3] K. Ito, *Proc. Intern. Congress Math.*, 15-22 VIII 1962, Uppsala, 1963, p. 536.

Note: Figure translations are in progress. See original paper for figures.