ON LEBESGUE SUBSPACES OF THE SPACE OF TRAJECTORIES OF A RANDOM PROCESS

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Abstract

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V. G. VINOKUROV, B. MAKHAMOV

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By a space with measure we shall mean here a collection $\{E, Q, m\}$, where E is a space, Q is a σ-algebra of measurable subsets of E, and m is a nonnegative countably additive measure defined on Q, such that the following conditions are satisfied: 1) $mE = 1$; 2) for any two points $x, y \in E$ there exists a set $A \in Q$ such that $x \in A,\ y \notin A$; 3) if $B \subset A$, where $A \in Q$ and $mA = 0$, then $B \in Q$.

Let E' be a subset of E of outer measure one. The space with measure $\{E', Q', m'\}$, where $Q' = Q \cap E'$, and $m'(A \cap E') = mA$, will be called a subspace of the space $\{E, Q, m\}$.

Consider the problem: which spaces with measure have a subspace that is a Lebesgue space. (For the definitions of a Lebesgue space and other measure-theoretic terms, see ([1]).)

If a space with measure has a Lebesgue subspace but itself is not a Lebesgue space, then this space cannot be separable.

We shall call a measurable partition ξ of the space $\{E, Q, m\}$ a dense partition if, for every measurable set A, there exists a measurable set B, which is a union of elements of the partition ξ, such that

$$m[(A \setminus B) \cup (B \setminus A)] = 0.$$

Any two dense partitions coincide $(\operatorname{mod} 0)$.

Theorem 1. *From the existence in the space $\{E, Q, m\}$ of Lebesgue subspaces there follows the existence in $\{E, Q, m\}$ of a dense partition. Every Lebesgue subspace intersects each element of the dense partition in one point* $(\operatorname{mod} 0)$*. If the space $\{E, Q, m\}$ is a product of Lebesgue spaces, then from the existence in it of a dense partition there follows the existence of Lebesgue subspaces.*

Let $\{\xi_t\}$ be a random process, where the time t ranges over some subset T of the real line. Denote by $\{E, Q, m\}$ the space of all possible trajectories (x_t) of this process. The σ-algebra Q and the measure m on it are defined in accordance with Kolmogorov' s well-known theorem on the extension of consistent finite-dimensional distributions. In many cases it is inconvenient to take, as the trajectory space of the process, the space of all possible trajectories, since then many important sets turn out to be nonmeasurable. Therefore, as the trajectory space one chooses some subspace $\{E', Q', m'\}$ of the space $\{E, Q, m\}$. It is customary to choose the subspace E' in accordance with the principle of separability of the process, introduced by Doob (see ([2])). The process $\{\xi_t\}$ is called separable relative to the subspace $\{E', Q', m'\}$ if there exist a countable subset $T' \subset T$ and a set N of measure zero such that, for every segment $[a, b]$ of the real line and every interval α,

$$\{(x_t):\ a \le x_t \le b,\ t \in \alpha \cap T\} \cap (E' \setminus N)$$
$$= \{(x_t):\ a \le x_{t'} \le b,\ t' \in \alpha \cap T'\} \cap (E' \setminus N).$$

Doob showed that for every stochastic process $\{\xi_t\}$ there exists a subspace $\{E', Q', m'\}$ with respect to which this process is separable (see (2)). However, the principle of separability does not affect the measure-theoretic properties of the subspace $\{E', Q', m'\}$. It can be shown that for every stochastic process there exist both subspaces with a complete measure and subspaces with a noncomplete measure with respect to which this process is separable.

If the set of all discontinuous trajectories of the process has outer measure one, then the subspace $\{E', Q', m'\}$, where E' is the set of all discontinuous trajectories, is a Lebesgue space and the process is separable with respect to it. In the general case, the following assertion can be made about the existence of Lebesgue subspaces with respect to which the process is separable.

Let us call the metric structure of the process $\{\xi_t\}$ the metric structure of the space of its trajectories $\{E, Q, m\}$, i.e. the set of classes of coinciding (mod 0) measurable sets $\{E, Q, m\}$, with the metric introduced on this set

$$\rho(A, B) = m[(A \setminus B) \cup (B \setminus A)].$$

The space $\{E, Q, m\}$ has a dense partition if and only if the metric structure of the stochastic process is separable. In particular, the metric structure of stochastically continuous processes is separable.

A dense partition of the space $\{E, Q, m\}$ is a partition whose elements are the sets

$$\{(x_t):\quad x_{t'} = c_{t'},\ t' \in T'\},$$

where $c_{t'}$ are fixed numbers, and T' is a countable subset of T, for which the cylinder sets corresponding to finite sets of coordinates from T' belong to classes forming a dense set in the metric structure.

Theorem 2. *A stochastic process has a Lebesgue subspace of the space of trajectories if and only if the metric structure of the process is separable. In this case the Lebesgue subspace can be chosen so that the process is separable with respect to it.*

Suppose that for the stochastic process $\{\xi_t\}$ the set T is Lebesgue measurable. Take the direct product of the set T, regarded as a measure space, and the subspace $\{E', Q', m'\}$. The process $\{\xi_t\}$ is called measurable with respect to the subspace $\{E', Q', m'\}$ if the function $\varphi(x_t, \tau) = x_\tau$ $(x_t \in E',\ \tau \in T)$ is measurable on this direct product (see (2)). It can be shown that if the metric structure of a stochastic process is separable and there exists a subspace of the space of trajectories with respect to which this process is measurable, then there also exists a Lebesgue subspace with respect to which this process is measurable and separable.

Institute of Mathematics named after V. I. Romanovskii
Academy of Sciences of the Uzbek SSR

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REFERENCES

1. V. A. Rokhlin, *Mat. sborn.*, **25** (67), 107 (1949).

2. J. L. Doob, *Stochastic Processes*, Moscow, 1956.

Note: Figure translations are in progress. See original paper for figures.