

# ON THE IDENTITY PROBLEM FOR FINITELY PRESENTED SEMIGROUPS

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## **ON THE IDENTITY PROBLEM FOR FINITELY PRESENTED SEMIGROUPS**

*(Presented by Academician P. S. Novikov, 13 IV 1967)*

Let the semigroup  $\Pi$  be given by generators

$$a_1, a_2, \dots, a_n \quad (1)$$

and defining relations

$$A_i = B_i \quad (\text{the words } A_i \text{ and } B_i \text{ are nonempty, } i = 1, 2, \dots, k). \quad (2)$$

The set  $M = \{A_i\} \cup \{B_i\}$  ( $i = 1, 2, \dots, k$ ) is called the **basis** of the semigroup  $\Pi$ . The beginning (end)  $P$  of a word  $X$  will be called a **proper beginning** (**proper end**) of the word  $X$ , if  $\partial(P) \geq \frac{1}{2}\partial(X)$ .

We shall say that the semigroup  $\Pi$  with generators (1) and defining relations (2) belongs to the class  $K_{1/2}$ , if its basis  $M$  satisfies the following conditions: a) if  $A, B \in M$ ,  $B \stackrel{\circ}{=} RPQ$ , and  $P$  is a proper beginning of the word  $A$ , then  $R$  is empty; b) if  $A, B \in M$ ,  $B \stackrel{\circ}{=} RPQ$ , and  $Q$  is a proper end of the word  $A$ , then  $Q$  is empty.

**Theorem 1.** *If  $\Pi \in K_{1/2}$ , then for every word  $Z$  in the alphabet (1) there exists only a finite number of words equal to it in  $\Pi$ .*

From Theorem 1 it follows easily that

**Theorem 2.** *There exists an algorithm that solves the identity problem for all semigroups of the class  $K_{1/2}$ .*

The same result for a substantially narrower class of semigroups was obtained by E. I. Grindlinger (see (2)).

By  $\overline{K}_{1/2}$  we denote the class of semigroups obtained analogously to the class  $K_{1/2}$ , if in the definition of proper beginnings and proper ends one considers, instead of the non-strict inequality, the strict inequality  $\partial(P) > \frac{1}{2}\partial(X)$ .

It is interesting to note that in the class  $\overline{K}_{1/2}$  one can indicate an example of a finitely presented semigroup with an unsolvable identity problem. Such examples are easily constructed on the basis of the known result of G. S. Tseitin <sup>(4)</sup>. In particular, the semigroup given by the generators  $a, b, c, d, e, m, n, p, q, m', n', y', q', \alpha, \beta, \gamma, \delta, \eta, \theta$  and defining relations

$$\begin{aligned} ac = ca, \quad ad = da, \quad bc = cb, \quad bd = db, \quad e = mn, \quad nc = cp, \quad mc = cq, \\ qpa = e, \quad e = m'n', \quad n'd = dp', \quad m'd = dq', \quad q'p'b = e, \quad ca = \alpha\beta\gamma, \\ c\alpha = \delta\theta, \quad cc = \delta\eta, \quad e = \alpha\gamma, \quad \eta aa = \theta\beta, \end{aligned} \quad (3)$$

belongs to the class  $\overline{K}_{1/2}$ , and in it the identity problem is unsolvable. An analogous example is indicated in the paper of E. I. Grindlinger <sup>(3)</sup>.

We proceed to the proof of Theorem 1. Let  $\Pi \in K_{1/2}$ . A word

$$E \stackrel{\circ}{=} S_1 S_2 \dots S_t \quad (4)$$

in the alphabet (1) will be called **normal** if the factors  $S_1, S_2, \dots, S_t$  satisfy the conditions:

- I. Each  $S_i$  is contained in some  $C_i \in M$ .
- II.  $S_1$  is a proper beginning of  $C_1$ ;  $S_t$  is a proper end of  $C_t$ .
- III. For each  $i < t$ , either  $S_i$  is a proper end  $C_i$ , or  $S_{i+1}$  is a proper beginning  $C_{i+1}$ .

Every decomposition (4) with properties I, II, and III will be called a **normal decomposition** of the word  $E$ .

**Lemma 1.** *If the word  $PAQ$ , where  $A \in M$ , has a normal decomposition consisting of  $t$  factors, then one can indicate such a normal decomposition (4) of the word  $PAQ$  that  $P \stackrel{\circ}{=} S_1 S_2 \dots S_{j-1}$ ,  $A \stackrel{\circ}{=} S_j$ .*

**Proof.** Let  $PAQ \stackrel{\circ}{=} R_1 R_2 \dots R_t$  be a normal decomposition, where  $C_\xi \stackrel{\circ}{=} u_\xi R_\xi q_\xi$ ,  $C_\xi \in M$  ( $\xi = 1, 2, \dots, t$ ). Obviously, one can indicate such an  $i \leq t$  that  $P \stackrel{\circ}{=} R_1 R_2 \dots R'_i$ ,  $R_i \stackrel{\circ}{=} R'_i A'$ ,  $A \stackrel{\circ}{=} A' A''$ , where  $A'$  is nonempty. Two cases are possible: 1)  $\partial(A') \geq \frac{1}{2}\partial(A)$ ; 2)  $\partial(A') < \frac{1}{2}\partial(A)$ .

**Case 1.** In this case  $A'$  is a proper beginning of the word  $A$ ;  $C_i \stackrel{\circ}{=} u_{iR} i' A' q_i$ , and, by condition a),  $u_{iR} i'$  is empty and  $R_i \stackrel{\circ}{=} A'$ . If in addition  $A''$  is empty, then  $R_i \stackrel{\circ}{=} A$ . Then put  $S_1 \stackrel{\circ}{=} R_1, \dots, S_{i-1} \stackrel{\circ}{=} R_{i-1}, S_i \stackrel{\circ}{=} A, S_{i+1} \stackrel{\circ}{=} R_{i+1}, \dots, S_t \stackrel{\circ}{=} R_t$  ( $\stackrel{\circ}{=} \stackrel{\circ}{\neq}$  denotes equality by definition). Obviously, the required decomposition (4) has been found.

Let  $A''$  be nonempty. Then  $R_i$  is not a proper end; indeed, suppose  $R_i$  is a proper end. Then  $q_i$  is empty and  $C_i \stackrel{\circ}{=} A'$ , i.e.  $A'$  is a proper end  $C_i$ , and

then  $A'$  is an end of  $A$ , i.e.  $A''$  is empty, contrary to the assumption. From condition III it follows that  $R_{i+1}$  is a proper beginning. If  $A'' \overset{\circ}{=} R_{i+1}Z_1$  for some  $Z_1$ , then  $R_{i+1}$  must be a beginning of  $A$ , which is impossible, since  $A'$  is nonempty. Therefore  $R_{i+1} \overset{\circ}{=} A''R_{i+1}'$ , where  $R_{i+1}''$  is nonempty. Note that if  $R_{i+1}$  is at the same time also a proper end, then  $R_{i+1}''$  is a proper end  $C_{i+1}$ . Put  $S_1 \rightleftharpoons R_1, \dots, S_{i-1} \rightleftharpoons R_{i-1}, S_i \rightleftharpoons A, S_{i+1} \rightleftharpoons R_{i+1}'', S_{i+2} \rightleftharpoons R_{i+2}, \dots, S_t \rightleftharpoons R_t$ , and the assertion of Lemma 1 is fulfilled for  $j = i$ .

**Case 2.** We have  $\partial(A'') > \frac{1}{2}\partial(A)$ , consequently  $A''$  is a proper end of  $A$ . If for some  $Z_2$  we have  $R_{i+1} \overset{\circ}{=} A''Z_2$ , then  $Z_2$  is empty, since  $A''$  must be an end of  $C_{i+1}$ . Therefore one may suppose that  $A'' \overset{\circ}{=} R_{i+1}Z_3$  for some  $Z_3$ .

Since  $A'$  is nonempty,  $R_{i+1}$  is not a proper beginning. Indeed, suppose  $R_{i+1}$  is a proper beginning. Then, since  $A \overset{\circ}{=} A'R_{i+1}Z_3$ ,  $R_{i+1}$  is a beginning of  $A$ , i.e.  $A'$  is empty, contrary to the assumption. From condition III it follows that  $R_i$  is a proper end. If at the same time  $R_i'$  is empty, then  $R_i \overset{\circ}{=} A'$ , and  $A'$  must be an end of  $A$ . But this is impossible, since  $A''$  is nonempty. Hence  $R_i'$  is nonempty. At the same time, if  $R_i$  was also a proper beginning, then  $R_i \overset{\circ}{=} C_i$ . Then  $R_i'$  is a proper beginning for  $C_i$ , since  $\partial(A) < \frac{1}{2}\partial(C_i)$ .

If  $R_{i+1}$  is a proper end, then  $Z_3$  is empty. Then put  $S_1 \rightleftharpoons R_1, \dots, S_{i-1} \rightleftharpoons R_{i-1}, S_i \rightleftharpoons R_i', \dots, S_{i+1} \rightleftharpoons A, S_{i+2} \rightleftharpoons R_{i+2}, \dots, S_t \rightleftharpoons R_t$ , and for  $j = i+1$  the assertion of Lemma 1 is fulfilled.

Let  $R_{i+1}$  not be a proper end. Then  $R_{i+2}$  is a proper beginning. If  $Z_3 \overset{\circ}{=} R_{i+2}Z_4$  for some  $Z_4$ , then  $R_{i+2}$  must be a beginning of  $A$ , but this is impossible. Consequently, we have  $R_{i+2} \overset{\circ}{=} Z_3R_{i+2}'$ , where  $R_{i+2}''$  is nonempty. Note that if  $R_{i+2}$  is at the same time also a proper end, then  $R_{i+2} \overset{\circ}{=} C_{i+2}$ . Then  $R_{i+2}'$  is a proper end  $C_{i+2}$ , since  $\partial(Z_3) < \frac{1}{2}\partial(C_{i+2})$ . The latter follows from the fact that in the word  $A \overset{\circ}{=} A'R_{i+1}Z_3$ ,  $Z_3$  is not a beginning. Put  $S_1 \rightleftharpoons R_1, \dots, S_{i-1} \rightleftharpoons R_{i-1}, S_i \rightleftharpoons R_i', S_{i+1} \rightleftharpoons A, S_{i+2} \rightleftharpoons R_{i+2}', S_{i+3} \rightleftharpoons R_{i+3}, \dots, S_t \rightleftharpoons R_t$ ,  $j = i+1$ . Lemma 1 is proved.

From Lemma 1 it follows immediately that

**Lemma 2.** *If  $PAQ$  has a normal decomposition into  $t$  factors, and  $PAQ \rightarrow PBQ$  is an elementary transformation of the semigroup  $\Pi$  (see (1), p. 7), then  $PBQ$  also has a normal decomposition into  $t$  factors.*

Let us call a normal word  $V$  the **maximal normal beginning** of the word  $VQ$ , if no word  $VQ'$ , with nonempty  $Q_1$  and  $Q \overset{\circ}{=} Q_1Q_2$

is not a normal word. The following lemma follows directly from Lemma 2.

**Lemma 3.** *Let  $V$  be the maximal normal beginning of the word  $VQ$ , and let  $V \rightarrow V_1$  be an elementary transformation of  $\Pi$ . Then  $V_1$  is the maximal normal beginning of the word  $V_1Q$ .*

**Lemma 4.** *If  $V$  is the maximal normal beginning of the word  $VY$  and  $VY \overset{\circ}{=} V_1Y_1$*

$PAQ$ , where  $A \in M$ , then either  $\partial(V) < \partial(P)$ , or  $\partial(V) \geq \partial(PA)$ .

**Proof.** Suppose that  $\partial(P) \leq \partial(V) < \partial(PA)$ . Then we have  $V \overset{\circ}{=} PQ'$ ,  $A \overset{\circ}{=} A'A''$ ,  $Y \overset{\circ}{=} A''Q$ , where  $A''$  is nonempty. We shall prove that  $VA''$  is a normal word. Let  $R_1 \dots R_t$  be the normal decomposition of the word  $V$ . It is enough to prove that  $A''$  is a proper end. Suppose this is not so. Then  $A'$  is a proper beginning.

Either  $R_t$  is an end of the word  $A'$ , or  $A'$  is an end of  $R_t$ . Arguing as in the proof of Lemma 1, it is easy to verify that both these cases are impossible. Consequently,  $VA''$  is a normal beginning of  $VY$ . This contradicts the fact that  $V$  is the maximal normal beginning of  $VY$ , since  $A''$  is nonempty. Lemma 4 is proved.

**Lemma 5.** *If  $\partial(X) = l$  and  $X = Y$  in  $\Pi$ , then  $\partial(Y) \leq lm^l$ , where  $m = \max\{\partial(A_i), \partial(B_i)\}$ .*

We shall prove this assertion by induction on the length of the word  $X$ .

Let  $\partial(X) = 0$ , i.e.,  $X$  is the empty word. If  $X = Y$  in  $\Pi$ , then  $Y$  is the empty word, since the defining relations (2) consist of nonempty words. Therefore,  $\partial(Y) = 0$ .

Let  $\partial(X) = l > 0$ . Then  $X \overset{\circ}{=} aW$ , where  $a$  is one of the generators (1). By the induction hypothesis, the length of any word  $W_j$  equal to  $W$  in  $\Pi$  does not exceed  $(l-1)m^{l-1}$ . Let  $Y \overset{\circ}{=} aW'$ . Among all sequences of elementary transformations  $aW_j \overset{\circ}{=} F_0 \rightarrow \dots \rightarrow F_q \overset{\circ}{=} Y$ , for various words  $W_j$  equal to  $W$ , choose a sequence

$$aW_k \overset{\circ}{=} E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E'_i \rightarrow E_{i+1} \rightarrow \dots \rightarrow E_2 \overset{\circ}{=} Y, \quad (5)$$

containing the smallest number of transformations. Let  $aW_k \overset{\circ}{=} VQ$ , where  $V$  is the maximal normal beginning of  $VQ$ .

If  $V$  is empty, then the first elementary transformation in (5) has the form  $P_1AP_2 \rightarrow P_1BP_2$ , where  $P_1$  is nonempty. Then  $P_1 \overset{\circ}{=} aP'_1$ . Put  $W_s \overset{\circ}{=} P'_1AP_2$ . Obviously  $W_s = W_k = W$  in  $\Pi$ , and the sequence  $aW_s \overset{\circ}{=} E_1 \rightarrow E_2 \rightarrow \dots \rightarrow E_r \overset{\circ}{=} Y$  is shorter than (5), which contradicts the condition by which the sequence (5) was chosen. Consequently, if  $V$  is empty, then  $Y \overset{\circ}{=} aW_k$ , and

$$\partial(Y) \leq 1 + (l-1)m^{l-1} \leq lm^l.$$

If  $V$  is nonempty, then  $V \overset{\circ}{=} aV'$  and  $W_k \overset{\circ}{=} V'Z$ . By induction on the number  $i$  of the word  $E_i$ , we shall prove that  $E_i \overset{\circ}{=} V_iZ$ , where  $V_i$  is the maximal normal beginning of  $E_i$ , and that there exists a sequence

$$aV' \overset{\circ}{=} V_0 \rightarrow V_1 \rightarrow \dots \rightarrow V_i, \quad (6)$$

containing exactly  $i$  elementary transformations. The word  $E_0$  has the indicated form. Suppose that our assertion is true for  $E_t$ , and let us prove it for  $E_{t+1}$ . Let the elementary transformation  $E_t \rightarrow E_{t+1}$  have the form  $PAQ \rightarrow PBQ$ , where  $A, B \in M$ . By Lemma 4, either  $\partial(V_t) < \partial(P)$ , or  $\partial(V_t) \geq \partial(PA)$ . We shall prove that the first case is impossible. Indeed, in this case we have  $P \overset{\circ}{=} V'_tP$  and  $Z \overset{\circ}{=} P'AQ$ . Then  $W_k \overset{\circ}{=} V'P'BQ$  in  $\Pi$ , whence  $V'P'BQ = W$  in  $\Pi$ . With the aid of the sequence (6), it is easy to construct the sequence of elementary transformations

$$aV'P'BQ \rightarrow V_1P'BQ \rightarrow \dots \rightarrow V'_{tP}BQ \overset{\circ}{=} E_{t+1} \rightarrow E_{t+2} \rightarrow \dots \rightarrow E_r,$$

which is shorter than the sequence (5).

In the second case we have  $V_t \overset{\circ}{=} PAV'_t$  and  $Q \overset{\circ}{=} V'_tZ$ . Then  $E_{t+1} \overset{\circ}{=} PBV'_tZ$ , and the transition  $PAV'_t \rightarrow PBV'_t$  is an elementary transformation of  $\Pi$ . By Lemma 3,  $PBV'_t$  is the maximal normal beginning of the word  $PBV'_tZ$ , i.e.  $PBV'_t$  is the required maximal normal beginning of  $V_{t+1}$ . Consequently,  $Y \circ V_{rZ}$ , where  $V_r = V$  in  $\Pi$ . Then

$$\begin{aligned} \partial(Y) &= \partial(V_r) + \partial(Z) \leq m[1 + (l-1)m^{l-1} - \partial(Z)] + \partial(Z) = \\ &= \partial(Z)(1-m) + lm^l + m(1-m)^{l-1} \leq lm^l. \end{aligned}$$

Lemma 5 is proved.

The assertion of Theorem 1 follows directly from Lemma 5.

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