# EXTENSION OF RECURSIVE HIERARCHIES AND \(R\)-OPERATIONS

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## Abstract

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**MATHEMATICS**

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# EXTENSION OF RECURSIVE HIERARCHIES AND $R$-OPERATIONS

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Here we consider extensions of recursive hierarchies that arise as a result of going beyond the constructive second class of ordinal numbers. For example, the $\mathfrak{H}$-hierarchy$^*$, whose finite levels were defined in (8), was extended in (2) on the basis of the constructive first, second, and third number classes and continued still further in (10) on the basis of a constructive version of higher classes of ordinal numbers. To estimate such extensions of recursive hierarchies we apply the $R$-operations$^*$ of A. N. Kolmogorov, which makes it possible to obtain estimates more precise than in (10) and (4). Analogously to the finite levels of the classical hierarchy of $R$-sets (11), we define a sequence of classes $\mathfrak{R}_n$ ($n = 0, 1, 2, ...$) of sets of natural numbers and establish (where $C\mathfrak{R}_n$ denotes the class of complements of sets of the class $\mathfrak{R}_n$): 1) the $\mathfrak{H}$-hierarchy (10) and the $H_h$-hierarchy (4) do not go beyond the class $\mathfrak{R}_2 \cap C\mathfrak{R}_2{}^{**}$; 2) in each of the classes $\mathfrak{R}_{n+1} \cap C\mathfrak{R}_{n+1}$ there is a transfinite hierarchy of sets not belonging to $\mathfrak{R}_n \cup C\mathfrak{R}_n$; 3) all $\mathfrak{R}_n \subset \Sigma^1_2 \cap \Pi^1_2$ (in (4, 10) the class $\Sigma^1_2 \cap \Pi^{1*}_2$ is indicated as an estimate for the sets covered by the $\mathfrak{H}$- and $H_h$-hierarchies).

**1.** ***R*-operations.** $N = \{0, 1, 2, ...\}$; $a, b, i, m, n$ are variables on $N$; $N^N$ is the set of all functions $f : N \to N$; $\mathcal{P}N$ is the set of all subsets of $N$; $\xi, \eta$ are variables on $\mathcal{P}N$. Subsets of $\mathcal{P}N$ will be called **bases**. The $\delta s$-operation $\Phi_M$ with base $M$ is the function which, to each sequence $\{E_i\}_{i\in N}$ of sets of an arbitrary space $E$ (in our case always $E = N$), assigns the set

$$\Phi_M\{E_i\}_i = \{x : (\exists\xi \in M)(\forall i \in \xi)[x \in E_i]\}.$$

The **completion** of a base $M$ is called the base

$$\widetilde{M} = \{\xi : (\exists\eta \in M)[\eta \subseteq \xi]\}.$$

For equality of the $\delta s$-operations $\Phi_{M_1}$ and $\Phi_{M_2}$ it is necessary and sufficient that $\widetilde{M}_1 = \widetilde{M}_2$.

For any base $M$, let

$$M^c = \{\xi : (N \setminus \xi) \notin M\}$$

and $M' = (\widetilde{M})^c$; then

$$\Phi_{M'}\{E_i\} = \{x : (\forall \xi \in M)(\exists i \in \xi)(x \in E_i)\}.$$

Let all tuples (i.e., finite ordered sets) of natural numbers be enumerated one-to-one by all natural numbers, with 0 the number of the empty tuple, and $[m_0, \dots, m_i]$ the number of the tuple $m_0, \dots, m_i$. Let

$$\theta(0, n) = [n]$$

and

$$\theta(m, n) = [m_0, \dots, m_i, n] \quad \text{for } m = [m_0, \dots, m_i];$$

we assume the enumeration of tuples is such that $\theta(m, n)$ is a general recursive function (g.r.f.). Let

$$\theta_m(n) = \theta(m, n)$$

and

$$\theta_m^{-1}(\xi) = \{n : \theta_m(n) \in \xi\}.$$

For any base $M$ define the base $\mathbf{R}(M)$ as follows:

$$\mathbf{R}(M) = \{\xi : 0 \in \xi \,\&\, (\forall m \in \xi)[\theta_m^{-1}(\xi) \in M]\}.$$

Obviously,

$$\widetilde{M}_1 = \widetilde{M}_2 \Rightarrow \widetilde{\mathbf{R}}(M_1) = \widetilde{\mathbf{R}}(M_2)$$

($\widetilde{\mathbf{R}}(M)$ is the completion of $\mathbf{R}(M)$). The $\delta s$-operation $\Phi_K$ with base $K = \mathbf{R}(M)$ is called the $R$**-operation over** $\Phi_M$. (This definition corresponds to ([11]).)

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[*] The definition is given below.

[**] This also clarifies the connection between the $\mathfrak{H}$-hierarchy (extended as in ($^{10}$)) and the effective $R$-hierarchy; on this connection see ($^{2}$).

For any function $F : \mathscr{P}N \to \mathscr{P}N$ define the function $\mathbf{L}_F : \mathscr{P}N \to \mathscr{P}N$ as follows. First, for any $\xi \subseteq N$ and ordinal $\nu < \Omega$ ($\Omega$ is the least uncountable ordinal) define $\xi^\nu$: 1) $\xi^0 = \xi$; 2) $\xi^{\nu+1} = F(\xi^\nu)$; 3) if $\nu$ is a limit ordinal, then $\xi^\nu = \bigcap_{\mu<\nu} \xi^\mu$; then put

$$\mathbf{L}_F(\xi) = \bigcap_{\Omega<\nu} \xi^\nu.$$

**Lemma** (adapted from ($^{12}$)). Let $M \subseteq \mathscr{P}N$ and

$$F(\xi) = \{m : m \in \xi \& \theta_m^{-1}(\xi) \in M\}.$$

Then

$$\widetilde{\mathbf{R}}(M) = \{\xi : 0 \in \mathbf{L}_F(\xi)\}.$$

If $X \subseteq \mathscr{P}N$ (or $X \subseteq N$), then let, in accordance with (1), $X \in \Sigma_n^1 \cap \Pi_n^1$ mean that the predicate $\xi \in X$ (or, respectively, $i \in X$) is expressible in both $n$-functional-quantifier Kleene forms ($^{7}$).

**Theorem 1.** For any base $M$, if $\widetilde{M} \in \Sigma_2^1 \cap \Pi_2^1$, then also

$$\widetilde{\mathbf{R}}(M) \in \Sigma_2^1 \cap \Pi_2^1.$$

The $\Sigma_2^1$-form for $\widetilde{\mathbf{R}}(M)$ is obtained from the definition of $\mathbf{R}(M)$, and the $\Pi_2^1$-form from the lemma and Patnam' s method ($^{13}$). (If the classes of $\Sigma_2^1$- and $\Pi_2^1$-subsets of $\mathscr{P}N$ are numbered, for example, as in ($^{7}$), then one can find such general recursive functions $g_0(m,n)$ and $g_1(m,n)$ that, for any pair $m, n$ of numbers of the two forms of the base $\widetilde{M}$, the numbers $g_0(m,n)$ and $g_1(m,n)$ are numbers of the two forms of the base $\widetilde{\mathbf{R}}(M)$.)

Define a sequence of bases $R_n$ ($n = 0, 1, 2, \ldots$) as follows: $R_0 = \{N\}$ (i.e. $R_0$ contains only one element, $N$), $R_{n+1} = \mathbf{R}(R_n)$.

The $\delta s$-operation $\Phi_{R_n}$ with base $R_n$ is called an $R^n$-operation. Thus, an $R^0$-operation is the operation of intersection

$$\Phi_{R_0}\{E_i\} = \bigcap_{i \in N} E_i,$$

and an $R^{n+1}$-operation is an $R$-operation over $\Phi'_{R_n}$. From Theorem 1 we obtain

**Corollary**$^*$. For any $n \in N$,

$$\widetilde{R}_n \in \Sigma_2^1 \cap \Pi_2^1 \quad \text{and} \quad \widetilde{R}'_n \in \Sigma_2^1 \cap \Pi_2^1.$$

**2. The classes $\mathfrak{R}_n$.** A one-place partial recursive function (p.r.f.) with Gödel number (see (6)) $a$ is denoted by $\langle a \rangle$. By $\tau$ we denote the family (see (3)) of all recursively enumerable subsets of $N$, having the following numbering: for any $a \in N$, the member of the family $\tau$ with number $a$, $\tau a$, is the domain of definition of the p.r.f. $\langle a \rangle$.

The class of all sets $\xi$ ($\subseteq N$) representable in the form

$$\xi = \{m : (\exists \eta \in R_n)(\forall i \in \eta) P(m, i)\},$$

where $P(m, i)$ is some recursively enumerable predicate, will be called the class $\mathfrak{R}_n$ ($n = 0, 1, 2, \ldots$). $C\mathfrak{R}_n$ is the class of complements (with respect to $N$) of sets of the class $\mathfrak{R}_n$.

Obviously, the class $\mathfrak{R}_n$ consists of all sets obtained by an $R^n$-operation over all possible recursively enumerable sequences of members of the family $\tau$. We provide each of the classes $\mathfrak{R}_n$ with a certain numbering, defining the families $\rho_n$ as follows:

$$\rho_n = H_{\widetilde{R}_n} \tau,$$

where $H_{\widetilde{R}_n}$ is the standard operator with base $\widetilde{R}_n$ (3). Since any $\delta s$-operation $\Phi_M$ with base $M$ coincides with the set-theoretic operation $\Psi_{\widetilde{M}}$ with base $\widetilde{M}$, then, by the definition of the standard operator, the member of the family $\rho_n$ with number $a$,

$$\rho_n a = \Phi_{R_n} \{E_i\},$$

where

$$E_i = \begin{cases} \tau \langle a \rangle (i), & \text{for those } i \in N \text{ for which } \langle a \rangle (i) \text{ is defined,} \\ \emptyset, & \text{for the remaining } i \in N. \end{cases}$$

The families $\bar{\rho}_n$ are defined as follows: for any $a \in N$,

$$\bar{\rho}_n a = N \setminus \rho_n a.$$

Obviously, the totality of all members of the family $\rho_n$ coincides with $\mathfrak{R}_n$, and the totality of all members of $\bar{\rho}_n$ with $C\mathfrak{R}_n$.

**Theorem 2.** For any $n \in N$,

$$\mathfrak{R}_n \subset \mathfrak{R}_{n+1}, \qquad C\mathfrak{R}_n \subset \mathfrak{R}_{n+1} \quad \text{and} \quad \mathfrak{R}_n \neq C\mathfrak{R}_n.$$

**Proof** by means of the properties of $R$-operations established in (5).

* Cf. (11), Corollary 2 of Theorem 12. See also our remark after Theorem 3.

**Theorem 3.** For all $n \in N$, $\mathfrak{R}_n \subset \Sigma^1_2 \cap \Pi^1_2$.

**Proof** on the basis of the corollary to Theorem 1.

**Remark.** With the aid of the system of notations for ordinals of the first two constructive number classes (9), the sequence of bases $R_n$ and classes $\mathfrak{R}_n$ can be continued transfinitely, obtaining an effective analogue of the classical hierarchy of $R$-sets (11). As the basic space, instead of $N$ one may consider the space

$$N^{m,n} = (N^N)^m \times N^n \qquad (m, n = 0, 1, \dots;\ m + n > 0)\ (^1)\,,$$

taking the family of all effectively open sets of this space instead of $\tau$. The corollary of Theorem 1 and Theorems 2 and 3 can be extended to both these cases.

**3. Hierarchies in the class $\mathfrak{R}_2$.** Introduce the one-one recursive functions $\sigma(m,n)$, $(n)'$, and $(n)''$, establishing a one-to-one correspondence between $N^2$ and $N$. The function

$$\mathrm{hj} : \mathcal{P}N \to \mathcal{P}N,$$

which assigns to each $\xi \subseteq N$ the set

$$\mathrm{hj}(\xi) = \{m : (\forall f)\,(\mathfrak{R}_n) T_1^{\xi, f}(m, m, n)\},$$

is called a **hyperjump** (hyperjump, (4,13)). The hyperjump can be represented by means of the $\delta s$-operation $\Phi_{R'_1}$; with the aid of this representation and known properties of $R$-operations (5,12) one obtains

**Theorem 4.** If $\xi \in \mathfrak{R}_2 \cap C\mathfrak{R}_2$, then also $\mathrm{hj}(\xi) \in \mathfrak{R}_2 \cap C\mathfrak{R}_2$, and there exists a recursive function $g(m,n)$ such that, if

$$\xi = \rho_2 m = \rho_2 n,$$

then

$$\mathrm{hj}(\xi) = \rho_2(g(m,n))' = \rho_2(g(m,n))''.$$

The field of a binary relation $P(m,n)$ is the set

$$\{m : (\exists n)[P(m,n) \vee P(n,m)]\}.$$

An irreflexive transitive binary relation $m <_W n$ with field $W (\subseteq N)$ is called a **system of notations** (s.o.), if: a) $1 \in W$ and

$$(\forall n \in W)[n \neq 1 \Rightarrow 1 <_W n];$$

b) every nonempty subset of the field $W$ has a $<_W$-minimal element.

For any function $F : \mathcal{P}N \to \mathcal{P}N$ and s.o. $<_W$ with field $W$, denote by $\Pi(F, <_W)$ the class of all and only those functions $\mathfrak{F} : W \to \mathcal{P}N$ for each of which there exist partial recursive functions $p(a)$, $q(a,b)$, and $r(a,b)$ such that, if $a \in W$ and $a \neq 1$, then:

a) $p(a) <_W a$ and

$$\{b : q(a,b) \in \mathfrak{F}(p(a))\} = \{b <_W a\};$$

b)

$$\mathfrak{F}(a) = \{m : r(a,m) \in F(U_a)\},$$

where

$$U_a = \{\sigma(b,m) : q(a,b) \in \mathfrak{F}(p(a)) \ \& \ m \in \mathfrak{F}(b)\}.$$

Thus, if $\mathfrak{F} \in \Pi(F, <_W)$, then the value $\mathfrak{F}(a)$, for $a \in W$ and $a \neq 1$, is determined according to the scheme given in b), through the values $\mathfrak{F}(b)$ for $b <_W a$.

**Theorem 5.** If $\mathfrak{F} \in \Pi(\mathrm{hj}, <_W)$, where $<_W$ is an s.o. with field $W$, and if

$$\mathfrak{F}(1) \in \mathfrak{R}_2 \cap C\mathfrak{R}_2,$$

then there exists a recursive function $f(a)$ such that for any $a \in W$

$$\mathfrak{F}(a) = \rho_2(f(a))' = \rho_2(f(a))''.$$

**Proof** on the basis of the recursion lemma (4,10) and Theorem 4.

**Corollary.** Let an s.o. $<_h$ with field $D_h$ and a function

$$H_h : D_h \to \mathcal{P}N$$

be defined by (4). Then

$$H_h(x) \in \mathfrak{R}_2 \cap C\mathfrak{R}_2$$

for all $x \in D_h$.

Now consider the transfinite iteration of the hyperjump on the basis of the constructive version of the class of all attainable ordinals. Let an s.o. $<_{\widetilde{C}}$ with field $\widetilde{C}$, and for each $a \in \widetilde{C}$ subsystems $I_a$ ($I_a \subset <_{\widetilde{C}}$) with field $\operatorname{dom} I_a$ (interpreted as the constructive number class of index $a$) be defined as in (10). For each $a \in \widetilde{C}$ there is in $\widetilde{C}$ a unique element immediately following $a$, $2^a$. If $a \in \widetilde{C}$ and

$$(\forall n)[a \neq 2^n],$$

then $a = 3^b 5^n$, where $b <_{\widetilde{C}} a$ and $n$ is the Gödel number of such a partial recursive function that

$$(\forall i \in \operatorname{dom} I_b)[\,\langle n\rangle(i) <_{\widetilde{C}} a\,].$$

**Theorem 6.** There exists a recursive function $g(a)$ such that for any $a \in \widetilde{C}$

$$\sigma(I_a) = \rho_2(g(a))' = \rho_2(g(a))''.$$

**Proof** by means of Theorem 5.

For all $a \in \widetilde{C}$, the sets $\mathfrak{H}_a (\subseteq N)$ are defined as follows ($^{10}$): 1) $\mathfrak{H}_1 = N$; 2) $\mathfrak{H}_{2^a} = \mathrm{hj}(\mathfrak{H}_a)$, where $a \in \widetilde{C}$; 3) if $a \in \widetilde{C}$ and $a = 3^b 5^n$, then

$$\mathfrak{H}_a = \{m : (m)_1 \in \operatorname{dom} I_b \ \& \ (m)_0 \in \mathfrak{H}_{\langle n \rangle ((m)_1)}\}.*$$

* For $m > 0$, $(m)_i$ is the exponent of the $i$-th prime number in the representation of $m$ as a product of powers of primes; $(0)_i = 0$ (see ($^6$)).

From Theorem 6, with the aid of Theorem 4, we obtain

**Corollary.** For all $a \in \widehat{C}$, $\mathfrak{H}_a \in \mathfrak{R}_2 \cap C\mathfrak{R}_2$.

**4. Hierarchies in the classes $\mathfrak{R}_{n+1}$.** Theorem 5 can be generalized as follows.

**Theorem 7.** For any $n = 0, 1, 2, \ldots$: if $\mathfrak{F} \in \Pi(F, <_W)$, where $<_W$ is a c.o. with field $W$, and the function $F : \mathfrak{P}N \to \mathfrak{P}N$ is representable in the form

$$F(\xi) = \{m : (\forall \eta \in R_n)(\exists i \in \eta) P^{\xi,\eta}(m, i)\}$$

with some general-recursive $P^{\xi,\eta}$, and if $\mathfrak{F}(1) \in \mathfrak{R}_{n+1} \cap C\mathfrak{R}_{n+1}$, then there exists an o.r.f. $f(a)$ such that, for every $a \in W$,

$$\mathfrak{F}(a) = \rho_{n+1}(f(a))' = \overline{\rho}_{n+1}(f(a))''.$$

On the basis of this theorem, in each of the classes $\mathfrak{R}_{n+1} \cap C\mathfrak{R}_{n+1}$ it is easy to indicate (taking, for example, $P^{\xi,\eta}(m, i) \Longleftrightarrow T_1^{\xi,\eta}(m, m, i)$ and, as the c.o. $<_W$, the Kleene system $<_0$ ($^9$)) a transfinite sequence of sets that do not belong to $\mathfrak{R}_n \cup C\mathfrak{R}_n$ and have strictly increasing degrees of recursive unsolvability.

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*Note: Figure translations are in progress. See original paper for figures.*

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