ON DIRECTIONAL DIFFERENTIATION OF A MAXIMIN FUNCTION

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Abstract

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ON DIRECTIONAL DIFFERENTIATION OF A MAXIMIN FUNCTION

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In ([1,2]), for the solution of minimax problems the maximum function was studied, and directional differentiability was proved for the function

$$\varphi(X) = \min f(X, Y).$$

The use of directional differentiability of such functions makes it possible to develop effective methods for solving a number of problems (best-approximation problems, minimax problems, etc.).

For the study of some other problems (for example, pursuit problems in the theory of differential games) it may be useful to study a function of the form

$$\varphi(Z) = \max_X \min_Y f(X, Y, Z).$$

Below we present some results connected with the question of the differentiability of the function $\varphi(Z)$ in directions.

Although a finite-dimensional case is considered here, many of the results can be generalized to the infinite-dimensional case (just as the results obtained in ([1]) for the finite-dimensional case were extended in ([2]) to a more general case).

1°. Let $f(X, Y, Z)$ be a function continuous in X and Y and continuously differentiable in Z on $\Omega_X \times \Omega_Y \times \Omega_Z$, where $\Omega_X \subset E_n,\ \Omega_Y \subset E_m;\ \Omega_Z \subset E_p$.

On Ω_Z consider the function

$$\varphi(Z) \equiv \max_{X \in \Omega_X} \min_{Y \in \Omega_Y} f(X, Y, Z). \tag{1}$$

The sets $\Omega_X, \Omega_Y, \Omega_Z$ are bounded and closed.

Fix some $Z \in \Omega_Z$. Let $g \in E_p;\ g \neq 0$ be such that, for $\alpha \in [0, \alpha_0]$ ($\alpha_0 = \alpha_0(g) > 0$), the point $Z_\alpha = Z + \alpha g \in \Omega_Z$. Such a g will be called an admissible direction.

It is required to find the derivative of $\varphi(Z)$ in the direction g

$$\varphi'_Z(g) \equiv \lim_{\alpha \to +0} \frac{\varphi(Z + \alpha g) - \varphi(Z)}{\alpha}. \tag{2}$$

Introduce into consideration the sets $R(Z) \subset \Omega_X$ and $Q(X, Z) \subset \Omega_Y$:

$$R(Z) = \left\{ X \mid X \in \Omega_X;\ \min_{Y \in \Omega_Y} f(X, Y, Z) = \max_{X \in \Omega_X} f(X, Y, Z) \right\},$$

$$Q(X, Z) = \left\{ Y \mid Y \in \Omega_Y;\ f(X, Y, Z) = \min_{Y \in \Omega_Y} f(X, Y, Z) \right\}.$$

In the present section we shall assume that the sets $Q(X, Z)$ satisfy the following rather stringent condition:

Condition A. For a given Z, the set $Q(X, Z)$ depends continuously on X on the set $R(Z)$ ($Q(X', Z) \to Q(X, Z)$ as $X' \to X;\ X' \in \Omega_X;\ X \in R(Z)$).

This means that

$$\rho(Q(X', Z), Q(X, Z)) \equiv$$

$$\equiv \sup_{Y \in Q(X', Z)} \inf_{V \in Q(X, Z)} \|V - Y\| + \sup_{V \in Q(X, Z)} \inf_{Y \in Q(X', Z)} \|V - Y\| \longrightarrow 0, \qquad X' \to X. \tag{3}$$

We note that this condition is satisfied, for example, if $Q(X, Z)$ consists, for every $X \in R(Z)$, of a single point.

Theorem 1. If at the point $Z \in \Omega_Z$ condition A is satisfied, then the function $\varphi(Z)$ is differentiable in any admissible direction, and

$$\varphi'_Z(g) = \max_{X \in R(Z)} \min_{Y \in Q(X, Z)} (\partial f(X, Y, Z)/\partial Z, g).$$

2°. Let the function $f(X, Y, Z)$ be given and continuous on $E_n \times \Omega_X \times \Omega_Z$, where $\Omega_X \subset E_m$, $\Omega_Z \subset E_p$; Ω_Y, Ω_Z are bounded closed sets. Fix a point $Z \in E_p$. Consider the sets $R(Z) \subset E_n$ and $Q(X, Z)$; they were defined in 1°, only here already

$$R(Z) = \{X \mid X \in E_n;\ \min_{Y \in \Omega_Y} f(X, Y, Z) = \max_{X \in E_n} \min_{Y \in \Omega_Y} f(X, Y, Z)\}.$$

We shall assume that:

1. For the point $Z \in E_p$ there does not exist a sequence of points $\{X_j\}$, $\|X_j\| \to \infty$, such that $\Phi(X_i, Z) \to \sup_{X \in E_n} \Phi(X, Z)$, where

$$\Phi(X, Z) = \min_{Y \in \Omega_Y} f(X, Y, Z).$$

2. The function $f(X, Y, Z)$ is twice continuously differentiable with respect to X and Z on $E_n \times \Omega_Y \times \Omega_Z$ and is strictly concave with respect to X for any fixed $Y \in \Omega_Y,\ Z \in \Omega_Z$, and the components of the vector functions $\partial f(X, Y, Z)/\partial X$, $\partial f(X, Y, Z)/\partial Y$ and the elements of the matrices $\partial^2 f(X, Y, Z)/\partial X^2$, $\partial^2 f(X, Y, Z)/\partial Z^2$, $\partial^2 f(X, Y, Z)/\partial X \partial Z$ are bounded on $E_n \times \Omega_Y \times \Omega_Z$.

Let $g \in E_p$ be an admissible direction.

Theorem 2. If conditions 1 and 2 are satisfied at the point Z, then the function

$$\varphi(Z) = \max_{X \in E_n} \min_{Y \in \Omega_Y} f(X, Y, Z)$$

is differentiable in any admissible direction g, and

$$\varphi'_Z(g) \equiv \lim_{\alpha \to +0} \frac{1}{\alpha} [\varphi(Z + \alpha g) - \varphi(Z)] =$$

$$= \max_{X \in R(Z)} \max_{V \in E_n} \min_{Y \in Q(X,Z)} \left[(\partial f(X, Y, Z)/\partial X, V) + (\partial f(X, Y, Z)/\partial Z, g) \right]. \quad (4)$$

3°. Finally, let the function $f(X, Y, Z)$ be given and continuous on $\Omega_X \times \Omega_Y \times \Omega_Z$, where $\Omega_X \subset E_n$, $\Omega_Y \subset E_m$, $\Omega_Z \subset E_p$ are bounded closed sets, and let $f(X, Y, Z)$ be twice continuously differentiable on $\Omega_X \times \Omega_Y \times \Omega_Z$ and strictly convex with respect to X for any fixed $Y \in \Omega_Y,\ Z \in \Omega_Z$. The sets $R(Z)$ and $Q(X, Z)$ are defined as above.

Theorem 3. The function

$$\varphi(Z) \equiv \max_{X \in \Omega_X} \min_{Y \in \Omega_Y} f(X, Y, Z)$$

is differentiable in any admissible direction g, and

$$\varphi'_Z(g) =$$

$$= \max_{X \in R(Z)} \max_{V \in M_X(\Omega_X)} \min_{Y \in Q(X,Z)} \left[(\partial f(X,Y,Z)/\partial X, V) + (\partial f(X,Y,Z)/\partial Z, g) \right], \tag{5}$$

where $M_X(\Omega_X)$ is the cone of admissible directions in the broad sense of the word, constructed at the point X with respect to the set Ω_X (see (3)).

Remark 1. Formulas (3), (4), (5) are also valid in the case when the sets Ω_Y and Ω_Z are not bounded. In this case an additional condition must be imposed on the function $f(X,Y,Z)$: the sets $Q(X,Z)$ must be bounded, i.e., for some $\varepsilon > 0$ the set

$$\bigcup_{X \in R(Z),\, Z' \in S_\varepsilon(Z)} Q(X, Z'),$$

where $S_\varepsilon(Z) \subset \Omega_Z$ is the sphere of radius ε with center at the point Z, must be bounded.

Remark 2. In a similar way one can get rid of the boundedness of the set Ω_X in 1°.

Remark 3. In 2° and 3° the fulfillment of condition A was not required.

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Note: Figure translations are in progress. See original paper for figures.