A NEW METHOD FOR INVESTIGATING THE ISOCHRONICITY OF A SYSTEM OF DIFFERENTIAL EQUATIONS

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Abstract

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A NEW METHOD FOR INVESTIGATING THE ISOCHRONICITY OF A SYSTEM OF DIFFERENTIAL EQUATIONS

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1°. Let a system of two differential equations be given:

$$\frac{dx}{dt} = y + \sum_{j+l=2}^{\infty} b_{jl} x^j y^l, \qquad \frac{dy}{dt} = -x - \sum_{j+l=2}^{\infty} c_{jl} x^j y^l. \tag{1}$$

Assume that the series appearing on the right-hand sides of system (1) converge in some sufficiently small neighborhood of the origin.

It is known (^{1}) that for system (1) the origin is a singular point of center or focus type. System (1) is called isochronous if an arbitrary point $M(x, y)$, lying on an arbitrary integral curve of this system in a sufficiently small neighborhood of the origin, makes a complete revolution about the origin in time $t = 2\pi$. There are two basic methods for investigating the isochronicity of system (1). The first of them is based on the following theorem of A. P. Vorob' ev (^{2}).

Theorem 1. *In order that system (1) have an isochronous center at the origin, it is necessary and sufficient that there exist a holomorphic transformation*

$$u = x + \sum_{j+l=2}^{\infty} \alpha_{jl} x^j y^l, \qquad v = y + \sum_{j+l=2}^{\infty} \beta_{jl} x^j y^l, \tag{2}$$

which reduces system (1) to the form

$$du/dt = v, \qquad dv/dt = -u. \tag{3}$$

From the investigations of A. P. Vorob' ev it follows that in Theorem 1 the word "holomorphic" may be replaced by the word "formal."

The second method (^{3}) is based on passing in system (1) to polar coordinates.

Just as in (4) a method was developed for finding center conditions, in the present paper we give a new method for investigating the isochronicity of system (1), connected with the introduction of the complex variable $w = x+iy$ ($\bar{w} = x-iy$), based on Theorem 1; we present some sufficient criteria for the isochronicity of system (1) and apply the results obtained to this system when the right-hand sides contain homogeneous nonlinearities only of third degree.

2°. Let system (1) be given. We shall seek conditions for the existence of an isochronous center of this system at the origin. Introduce the variable $w = x+iy$. Then system (1) is replaced by one equation of the form (5)

$$i\frac{dw}{dt} = w + \sum_{r=2}^{\infty} \varphi_r(\bar{w}, w), \qquad \text{where } \varphi_r(\bar{w}, w) \equiv \sum_{j+l=r} 2^{-j-l} z_{jl} \bar{w}^j w^l. \tag{4}$$

According to Theorem 1, denoting $W = u + iv$, we shall seek a transforma- of the form

$$W = w + \sum_{r=2}^{\infty} \Phi_r(\bar{w}, w), \tag{5}$$

where

$$\Phi_r(\bar{w}, w) = \sum_{j+l=r} \gamma_{jl} \bar{w}^j w^l, \tag{6}$$

and γ_{jl} are complex numbers transforming equation (4) into the equation

$$i\, dW/dt = W. \tag{7}$$

If a transformation (5) with the indicated property exists, then the coefficients γ_{jl} must satisfy the following conditions:

$$\gamma_{20} = {}^1\!/_{12}\, z_{20}, \qquad \gamma_{11} = {}^1\!/_4\, z_{11}, \qquad \gamma_{02} = -{}^1\!/_4\, z_{02}, \tag{8}$$

$$2^{j+l}(j-l+1)\gamma_{jl} = z_{jl} + \sum_{p+q=2}^{j+l-1} 2^{p+q-1} q \left(\gamma_{pq} z_{j-p,\, l-q+1} - \gamma_{qp} \bar{z}_{l-p,\, j-q+1}\right) \tag{9}$$

$$(j+l = 3, 4, 5, \ldots).$$

For the origin of coordinates to be an isochronous center of system (1), it is necessary and sufficient that system (9) be consistent.

Analogously to how this was done by A. M. Lyapunov (1) for the case of a center, one can show that if system (9) is consistent, then it has a solution with arbitrarily prescribed in advance values of the unknowns $\gamma_{l-1,l}$ $(l=2,3,...)$; therefore in what follows we shall assume

$$\gamma_{l-1,l}=0 \qquad (l=2,3,...). \tag{10}$$

Putting $j=l-1$ in (9), we obtain the consistency conditions of this system in the form

$$z_{l-1,l}+\sum_{p+q=}^{2(l-1)} 2^{p+q-1}q\left(\gamma_{pq}z_{l-p-1,\,l-q+1}-\gamma_{qp}\bar{z}_{l-p,\,l-q}\right)=0. \tag{11}$$

Theorem 2. *Conditions (11) are necessary and sufficient for the origin of coordinates to be an isochronous center of system (1).*

In particular, as the first isochronicity condition of system (1), from (11) for $l=2$, taking (8) into account, we have

$$6z_{12}-3z_{11}z_{02}-3z_{11}\bar{z}_{11}-2z_{20}\bar{z}_{20}=0. \tag{12}$$

3°. We indicate several sufficient criteria for the isochronicity of system (1).

Theorem 3. The equation

$$i\,dw/dt=w+\sum_{k=2}^{\infty}2^{-k}z_{0k}w^k \tag{13}$$

has an isochronous center at the origin of coordinates.

Indeed, the presence of a center was proved in (6). From (13) we obtain

$$dt=i\,dw\Big/\left(w+\sum_{k=2}^{\infty}2^{-k}z_{0k}w^k\right)=i\left[\frac{1}{w}+\psi(w)\right]dw, \tag{14}$$

where $\psi(w)$ is an analytic function of w. Integrating relation (14) along a closed integral curve surrounding the origin of coordinates, we shall have $T=-2\pi$, which means precisely the isochronicity of the center.

Theorem 4. *If the conditions*

$$b_{j,l-1}=-c_{j-1,l} \qquad (j+l=3,4,...) \tag{15}$$

(Here b_{jl} and c_{jl} for which at least one index is negative are considered equal to zero) system (1) is isochronous.

Indeed, passing to polar coordinates in system (1) with conditions (15), we obtain $d\varphi/dt = -1$; hence, integrating with respect to φ from 0 to 2π, we shall have $T = -2\pi$, which means that system (1) is isochronous.

4°. As an application of the method set forth above, consider the differential equation

$$i\,dw/dt = w + \frac{1}{8}(z_{30}\bar{w}^3 + z_{21}\bar{w}^2 w + z_{12}\bar{w}w^2 + z_{03}w^3). \tag{16}$$

We shall suppose that in (16)

$$\mathrm{Re}(z_{21} - 3z_{03}) = 0. \tag{17}$$

This condition can be achieved by a certain rotation of the coordinate axes.

According to (7), when condition (17) is fulfilled, equation (16) has a center at the origin if and only if at least one of the following series of conditions is fulfilled:

$$\text{I.} \quad \mathrm{Im}\, z_{12} = \mathrm{Im}(3z_{03} - \bar{z}_{21}) = 0. \tag{18}$$

$$\text{II.} \quad z_{12} = z_{03} + 3\bar{z}_{21} = 4z_{21}\bar{z}_{21} - z_{30}\bar{z}_{30} = 0. \tag{19}$$

$$\text{III.} \quad \mathrm{Im}\, z_{12} = \mathrm{Re}\, z_{21} = \mathrm{Im}\, z_{30} = 0. \tag{20}$$

The first two isochronicity conditions for equation (16) give

$$z_{12} = 0, \tag{21}$$

$$4z_{21}z_{03} + 4z_{21}\bar{z}_{21} + 3z_{30}\bar{z}_{30} = 0. \tag{22}$$

These conditions are sufficient to establish that in cases (18) and (19) equation (16) cannot have an isochronous center at the origin unless it is linear.

Let conditions (20) be fulfilled. Then, taking (17) into account, we have

$$z_{21} + \bar{z}_{21} = z_{03} + \bar{z}_{03} = z_{30} - \bar{z}_{30} = 0. \tag{23}$$

Condition (22) then gives

$$4z_{21}z_{03} - 4z_{21}^2 + 3z_{30}^2 = 0. \tag{24}$$

The third and fourth isochronicity conditions (11) for $l = 4, 5$ for equation (16), taking (21) and (23) into account, have the form

$$z_{30}(z_{03} - 3z_{21})(3z_{03} - 7z_{21}) = 0, \tag{25}$$

$$-321z_{30}^2 z_{21} z_{03} - 18z_{30}^2 z_{03}^2 + 181z_{30}^2 z_{21}^2 - 120z_{21}^2 z_{03}^2 - 72z_{21} z_{03}^3 + 144z_{21}^3 z_{03} + 48z_{21}^4 = 0. \tag{26}$$

According to (25), the following cases are possible:

a) $z_{30} = 0$. Then from (23) and (24) we have that either $z_{12} = z_{21} = z_{30} = 0$, or $z_{12} = z_{30} = z_{21} + \bar{z}_{21} = z_{21} - z_{03} = 0$. In both cases the origin is isochronous on the basis of Theorems 3 and 4, respectively.

b) $z_{03} - 3z_{21} = 0$. From (21), (23), (24), and (26) we obtain that equation (16) is linear.

c) $3z_{03} - 7z_{21} = 0$. From (23) and (24) we obtain

$$z_{12} = z_{30} - \bar{z}_{30} = z_{21} + \bar{z}_{21} = 3z_{03} - 7z_{21} = 16z_{21}^2 + 9z_{30}^2 = 0. \tag{27}$$

We shall show that the equation obtained in this case,

$$i\,dw/dt = w + \frac{1}{24} z_{21}(\pm 4i\bar{w}^3 + 3\bar{w}^2 w + 7w^3), \tag{28}$$

with the condition $z_{21} + \bar{z}_{21} = 0$, is isochronous. Indeed, by the substitution $w_1 = (1 + i)w$ we reduce equation (28) to the form

$$i\,\frac{dw_1}{dt} = w_1 + \frac{1}{4} a(\pm 4\bar{w}_1^3 - 3\bar{w}_1^2 w_1 + 7w_1^3), \tag{29}$$

where a is a real number.

This equation corresponds to two systems, namely

$$dx/dt = y + 3ax^2 y, \qquad dy/dt = -x - 2ax^3 + 9axy^2; \tag{30}$$

$$dx/dt = y + 9ax^2 y - 2ay^3, \qquad dy/dt = -x + 3axy^2. \tag{31}$$

It is easy to verify that, by substitutions of the form (2),

$$u = (x + 2ax^3)/(1 + 3ax^2)^{3/2}, \qquad v = y/(1 + 3ax^2)^{3/2}, \tag{32}$$

and, respectively,

$$u = x/(1-3ay^2)^{3/2}, \qquad v = (y-2y^3)/(1-3ay^2)^{3/2}, \tag{33}$$

systems (30) and (31) are reduced to the form (3), and, according to Theorem 1, both systems are isochronous.

Thus, in order that equation (16), under condition (17), have an isochronous center at the origin, it is necessary and sufficient that at least one of the following three series of conditions be satisfied:

$$\text{I.} \quad z_{12} = z_{21} = z_{30} = 0. \tag{34}$$

$$\text{II.} \quad z_{12} = z_{30} = z_{21} + \bar{z}_{21} = z_{21} - z_{03} = 0. \tag{35}$$

$$\text{III.} \quad z_{12} = z_{30} - \bar{z}_{30} = z_{21} + \bar{z}_{21} = 3z_{03} - 7z_{21} = 16z_{21}^2 + 9z_{30}^2 = 0. \tag{36}$$

It is not difficult to observe that, under condition (17), at least one of the series of conditions (34), (35), (36) is fulfilled if and only if, under condition (17), at least one of the following series of conditions holds:

$$\text{I.} \quad z_{12} = z_{21} = z_{30} = 0. \tag{37}$$

$$\text{II.} \quad z_{12} = z_{30} = z_{03} + \bar{z}_{21} = 0. \tag{38}$$

$$\text{III.} \quad z_{12} = 3z_{03} + 7\bar{z}_{21} = 9z_{30}\bar{z}_{30} - 16z_{21}\bar{z}_{21} = z_{30}\bar{z}_{21}^2 - \bar{z}_{30}z_{21}^2 = 0. \tag{39}$$

Taking into account that conditions (37), (38), and (39) are invariant under a rotation of the coordinate axes (8), we arrive at the theorem:

Theorem 5. *In order that equation (16) have an isochronous center at the origin, it is necessary and sufficient that at least one of the three series of conditions (37), (38), (39) be satisfied.*

Passing from this to the variables x and y, we obtain

Theorem 6. *In order that the system*

$$dx/dt = y + (\psi+\theta)x^3 + (\chi-\beta)x^2y + (3\psi-3\theta-\gamma)xy^2 + (\mu-\nu)y^3, \tag{40}$$

$$dy/dt = -x - (\mu+\nu)x^3 - (3\psi+3\theta+\alpha)x^2y - (\chi+\beta)xy^2 - (\psi-\theta)y^3$$

have an isochronous center at the origin, it is necessary and sufficient that at least one of the following three series of conditions be satisfied:

$$\text{I.}\quad \mu = \chi = \theta = \alpha+\gamma = \beta+3\nu = \alpha+6\psi = 0.$$

$$\text{II.}\quad \mu = \chi = \theta-\nu = \alpha+\gamma = \alpha+4\psi = 0.$$

$$\text{III.}\quad \chi+3\mu = \alpha+\gamma = \beta+6\nu = 5\alpha+24\psi = 4\psi^2+25(\nu^2-\mu^2-\theta^2) = \theta\alpha^2-4\theta\beta^2-4\alpha\beta\mu = 0.$$

From Theorem 6 it follows that the work of N. A. Lukashevich (9) does not contain all cases of isochronicity of the center of system (40).

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Note: Figure translations are in progress. See original paper for figures.