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THEORY OF ELASTICITY AND THE DEFORMATION THEORY OF PLASTICITY FOR SOLIDS WITH DIFFERENT PROPERTIES IN COMPRESSION, TENSION, AND TORSION

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Abstract

Full Text

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THEORY OF ELASTICITY

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THEORY OF ELASTICITY AND THE DEFORMATION THEORY OF PLASTICITY FOR SOLIDS WITH DIFFERENT PROPERTIES IN COMPRESSION, TENSION, AND TORSION

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Let us consider a homogeneous solid which, in the case of elastoplastic deformation, has different properties in uniaxial tension, uniaxial compression, and torsion, under the condition that the material remains isotropic with respect to each type of stress state.

We describe such a medium in a rectangular Cartesian coordinate system $x_i,\ i = 1, 2, 3$. The elastoplastic strains e_{ij} are assumed small, and the time interval t of deformation is such that the phenomena of creep and stress relaxation σ_{ij} may be neglected.

In the case where elastoplastic deformation of the body takes place, the laws for proportional or nearly proportional loading are taken in the form

$$\sigma_{ij} = 2G_1(\beta)\psi(T/T_0)(1-\omega(\bar{e}_u))\left(e_{ij} - {}^1/_3 e_{kk}\delta_{ij}\right) + 3K_0(T/T_0, e_1/|e_1|)e_1\delta_{ij} \quad (i,j = 1,2,3), \tag{1}$$

$$3e_1 = e_{kk} - 3\alpha(T - T_0), \qquad e_u = \bar{e}_u e_{s1}(\gamma)\chi(T'/T_0), \qquad \beta(1+\gamma) = \gamma,$$

$$\gamma = e_1/e_u,$$

if

$$\bar{e}_u > 1, \qquad \partial_t(\bar{e}_u) > 0, \qquad e_u^2 = {}^2/_3\left(e_{ij} - {}^1/_3 e_{kk}\delta_{ij}\right)^2, \qquad \partial_t(\bar{e}_u) = \partial\bar{e}_u/\partial t.$$

If $\bar{e}_u \leq 1$, then the function $\omega(\bar{e}_u)$ is identically equal to zero and elastic deformation takes place under any loading. The functions ω, e_{s1}, G_1 must satisfy the conditions

$$0 \leq \omega \leq \omega + \bar{e}_u \partial^2_{\bar{e}_u} \omega < 1 - \delta^2, \qquad \partial_{\bar{e}_u} \omega > 0, \qquad 2\partial_{\bar{e}_u} \omega > -\bar{e}_u \partial^2_{\bar{e}_u} \omega, \qquad \delta^2 = \text{const} \tag{2}$$

and conditions (13), (14), (15), (16), (17). In formulas (1), the coefficient of linear expansion is denoted by α, and the initial temperature of the body, constant at all points of the body, is denoted by T_0. For this temperature, if no loads act on the body, the stress components at every point of the body are equal to zero.

The modulus $G_1(\beta)\psi(T'/T_0) > 0$ and the yield limit (limit of proportionality) $e_{s1}(\gamma)\chi(T'/T_0) > 0$ are determined from four experiments: uniaxial compression, tension, torsion, and uniaxial compression with all-round pressure. The function $\omega(\bar{e}_u)$ is determined from a uniaxial-compression experiment.

The laws of proportional unloading, or of unloading close to it, for $\partial_t \bar{e}_u \leq 0$ are taken in the form

$$\tilde{\sigma}_{ij} = \sigma^*_{ij} - \sigma^0_{ij} = 3G_1(\beta)\psi(T'/T_0)\left(\tilde{e}_{ij} - {}^1/_3 \tilde{e}_{kk}\delta_{ij}\right) + K_0(\tilde{e}_{kk} - 3\alpha T),$$
$$\tilde{e}_{ij} = e^*_{ij} - e^0_{ij}. \tag{3}$$

The components σ^0_{ij}, e^0_{ij} are reckoned from the moment at which unloading begins, marked by the superscript "asterisk." For a justification of the proposed

relations (1) in the case of proportional loading, we shall make use of the laws of nonequilibrium thermodynamics of a solid body that does not exchange mass with other systems and is chemically inert for the process of small deformation. Suppose that the value of the temperature T at any point of the body differs little from T_0.

In the case of elastic (reversible) deformation, the free energy φ is constructed, from which relations (1) follow for $\omega \equiv 0$.

Now let us consider the elastoplastic deformation of a body. Here, in addition to the two basic laws of thermodynamics, we shall use the hypotheses (4), which are a certain generalization of Onsager's principle.

Following (4), we represent the generalized forces (components of the stress tensor) as the sum of two terms

$$\sigma_{ij} = \sigma^{(o)}_{ij} + \sigma^{(\,)}_{ij},$$

where the increment of the reversible part of the work $W^{(o)}$ is determined by the formula

$$dW^{(o)} = \sigma_{ij}^{(o)}\, de_{ij}.$$

We shall assume that the irreversible part of the work $W^{(\,)}$ is determined by the relation

$$W^{(\,)} = D = \sigma_{ij}^{(\,)} \dot{\vartheta}_{ij} \geqslant 0,$$

where the components $\vartheta_{ij}^{(\,)}$ are equal to

$$\vartheta_{ij}^{(\,)} = \left(\frac{\partial D}{\partial \dot{\vartheta}_{nm}} \dot{\vartheta}_{nm} \right)^{-1} D \, \frac{\partial D}{\partial \dot{\vartheta}_{ij}}.$$

In order to be entitled to apply this formula, it is necessary that the system be stable, i.e., that $\vartheta_u^{(\,)}$ increase with increasing ϑ_u. We shall use this condition below.

We shall consider proportional loading. In this case we take the dissipative function D in the form

$$D = 2G_1(\beta)\psi_1(\bar{e}_u) e_u \sqrt{{}^3/{}_2 \dot{\vartheta}_{ij} \dot{\vartheta}_{ij}} = \sigma_{ij}^{(\,)} \dot{\vartheta}_{ij} = 3G_1(\beta)\psi_1(\bar{e}_u) e_u \dot{e}_u. \qquad (4)$$

Then the values of the reversible part of the stresses $\vartheta_{ij}^{(o)}$ as functions of the components ϑ_{ij} will be equal to

$$\vartheta_{ij}^{(o)} = \frac{\partial}{\partial \vartheta_{ij}} \int_0^{\sigma_u/3G} \sigma_u \, d\frac{\sigma_u(\vartheta_{ij})}{3G(\beta)}. \qquad (5)$$

Thus, the components of the stress tensor, according to formulas (4), (5), are equal to

$$\sigma_{ij} = \sigma_{ij}^{(\,)} + \sigma_{ij}^{(o)} = 2G(\beta)(\psi_1 + \chi_1)(e_{ij} - e\delta_{ij}) + 3K_0 e_1 \delta_{ij}, \qquad (6)$$

or, introducing the function $\omega(e_u)$, they have the form according to formulas (1). In formulas (6) the function χ_1 is introduced on the basis of (5).

Now let us compute the stress components $\vartheta_{ij}^{(o)}$ by formula (5), using the expression for the function σ_u according to formulas (1). Then

$$\vartheta_{ij}^{(o)} = \sigma_u \frac{\partial \sigma_u/3G}{\partial e_u} \frac{\partial e_u}{\partial \vartheta_{ij}} = 2G(1-\omega) \left[1 - \omega - e_u \frac{\partial \omega}{\partial e_u} \right] (e_{ij} - e\delta_{ij}). \qquad (7)$$

Hence, the irreversible stress components are equal to

$$\sigma_{ij}^{()} = \sigma_{ij} - \sigma_{ij}^{(o)} = \frac{2\sigma_u}{3e_u}\left(1 - \frac{\partial\sigma_u/3G}{\partial e_u}\right)(e_{ij} - e\delta_{ij}),$$

$$\sigma_u^{()} = \vartheta_u^{()} = \sigma_u\left[1 - \partial_{e_u}(\sigma_u/3G)\right]. \tag{8}$$

Now let us obtain the basic inequalities that follow from the thermodynamic analysis. These inequalities are obtained from the conditions:

a) the dissipative function is positive

$$D = \sigma_u[1 - \partial_{e_u}(\sigma_u/3G)]\dot{e}_u \geq 0; \tag{9}$$

b) the intensity of the irreversible stress components $\sigma_u^{()}$ increases with increasing intensity $\bar{e}_u$, i.e.

$$\partial_t\sigma_u^{()} = \dot{\sigma}_u^{()} > 0, \qquad \partial_t\bar{e}_u = \dot{\bar{e}}_u > 0, \qquad \bar{e}_u > 1. \tag{10}$$

In what follows we consider a medium for which the conditions are satisfied

$$-p = \frac{1}{e_s}\partial_\gamma e_s < 0, \qquad -m = \frac{1}{3G}\partial_\beta 3G < 0. \tag{11}$$

To ensure inequalities a) and b), it is sufficient that the following inequalities hold:

$$0 \leq \omega < \omega + \bar{e}_u\partial_{e_u}^{-}\omega < 1 - \delta^2, \qquad 2\partial_{e_u}^{-}\omega > e_u\partial_{e_u}^2\omega + \delta_2^2 > 0,$$

$$3G = 3G_1\psi(T/T_0); \tag{12}$$

$$\dot{e}_u > 0, \qquad \dot{\bar{e}}_u > 0, \qquad \bar{e}_u \geq 1; \tag{13}$$

$$1 > K_1 > \frac{\bar{e}_u p\gamma\partial_{e_u}^{-}\omega}{\omega + (1 + p\gamma)\bar{e}_u\partial_{e_u}^{-}\omega} + \left(1 + \frac{m(1-\gamma^2)}{p(1+\gamma^2)}\right)\frac{p\bar{e}_u\partial_t\gamma}{\partial_t\bar{e}_u}c_1 + c_2c_3 > 0; \tag{14}$$

$$K_2(1 - p\gamma) \geq c_3c_4\partial_{e_u}^{-}\omega > 0; \tag{15}$$

$$K_1 = \frac{1 - \omega - \bar{e}_u\partial_{e_u}^{-}\omega}{1 - \omega}, \qquad K_2 = 2\partial_{e_u}^{-}\omega + \bar{e}_u\partial_{e_u}^2\omega, \qquad c_4 = 1 + \gamma\partial_\gamma e_u p;$$

Fig. 1. Deformation curves for ZrO_2 P-47 in coordinates $\sigma/\sigma_s = f(e/e_s)$. *a*—compression, $T = 1600°$; *b*—tension, $T = 1600°$; *v*—compression, $T = 1400°$.

Figure 1: Fig. 1. Deformation curves for ZrO_2 P-47 in coordinates $\sigma/\sigma_s = f(e/e_s)$. *a*—compression, $T = 1600°$; *b*—tension, $T = 1600°$; *v*—compression, $T = 1400°$.

$$c_2 - 1 = \delta_3^2 \ll 1;$$

$$K_1 > c_2 c_3^0 \text{ for } \gamma < 0; \qquad c_3^0 = p\gamma/(1 + p\gamma); \tag{16}$$

$$c_4 - 1 = \delta_4^2 \ll 1 \quad \text{for } \gamma > 0. \tag{17}$$

Note that conditions (16) ensure that the strain components e_{ij} are uniquely expressed through the stress components σ_{ij}, both in the case of elastic-plastic and elastic deformations.

Conditions (12), (13), (14), (15), (16), (17) guarantee that, with increasing shear deformation referred to the yield-limit deformation, the stress intensity referred to the yield limit increases. We shall call such deformation active. It is accompanied by an increase in elastic-plastic deformations.

Fig. 1. Deformation curves for ZrO_2 P-47 in coordinates $\sigma/\sigma_s = f(e/e_s)$. *a*—compression, $T = 1600°$; *b*—tension, $T = 1600°$; *v*—compression, $T = 1400°$.

	σ_s, kg/cm^2	e_s, %	T, °C	$E \cdot 10^{-5}$, kg/cm^2
Compression	8	0.14	1600	0.12
Compression	4.6	0.23	1400	0.15
Tension	0.92	0.045	1600	0.07

We determine the temperature field independently of the deformation pattern for the case when the strain rate is small and the rate of input of the external heat flux is small.

We shall assume that laws (1), (3) are also valid for more significant gradients of the temperature field in the body. Indeed, in these materials the compression modulus is 30% greater than the tensile modulus, while the compressive yield limit is an order of magnitude greater than the tensile yield limit (Fig. 1).

At the same time, the hypotheses of similarity are also experimentally confirmed in the case of uniaxial compression and tension, i.e.,

$$\sigma_u/\sigma_s = \bar{\sigma}_u = \bar{e}_u[1-\omega(\bar{e}_u)], \qquad G = G_1(\beta)\psi(T/T_0),$$

$$e_s = e_{s_1}(\gamma)\chi(T/T_0).$$

Thus, the first relation means that the curves of uniaxial compression and tension, represented in the coordinates $\bar{\sigma}_u, \bar{e}_u$, coincide (Fig. 1).

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Note: Figure translations are in progress. See original paper for figures.