ASYMPTOTICS OF A PERIODIC SOLUTION OF A NEUTRAL-TYPE EQUATION WITH SMALL DELAY

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Abstract

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ASYMPTOTICS OF A PERIODIC SOLUTION OF A NEUTRAL-TYPE EQUATION WITH SMALL DELAY

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In recent years, in a number of papers by A. B. Vasil' eva and the author ([1–3]), the basic initial-value problem (see ([4])) for a neutral-type equation with small deviation of the argument has been studied rather thoroughly. In these works the behavior of the solution under unbounded decrease of the delay was clarified, and an algorithm was developed for constructing an asymptotic representation of the solution.

The present note is a continuation of these investigations for the case of a periodic solution. A similar problem, but under substantial restrictions, was considered by the author in ([5]).

For simplicity of exposition, the proofs will be carried out for the scalar case. The restrictions imposed in the vector case will be indicated at the end of the note.

Thus, consider the equation

$$\dot{x}(t) = f(t, x(t), x(t - h\tau(t)), \dot{x}(t - h\tau(t))), \tag{1}$$

where $h > 0$ is a small parameter and $\tau(t)$ is some function, whose sign is unimportant to us.

Putting $h = 0$ in (1), we obtain the degenerate equation

$$\dot{x}(t) = f(t, x(t), x(t), \dot{x}(t)). \tag{2}$$

Let us denote the arguments of the function f as follows: $f(t, x, y, u)$, and formulate the main result of the note in the form of two theorems.

Theorem 1. *Suppose that:*

1) there exists an ω-periodic solution $\bar{x}(t)$ of the degenerate equation (2);

2) the functions $f(t,x,y,u)$ and $\tau(t)$ are ω-periodic functions of t;

3) in a neighborhood of the degenerate solution $\bar{x}(t)$ the function $f(t,x,y,u)$ is twice continuously differentiable, and the function $\tau(t)$ is continuously differentiable;

4) along the degenerate solution

$$|\bar{f}_u| \equiv \left|\frac{\partial f}{\partial u}(t,\bar{x},\bar{x},\dot{\bar{x}})\right| \neq 1; \tag{3}$$

$$\int_0^\omega (1-\bar{f}_u)^{-1}(\bar{f}_x+\bar{f}_y)\,dt \neq 0. \tag{4}$$

Then, for small h, in a neighborhood of $\bar{x}(t)$ there exists a unique ω-periodic solution of equation (1), and moreover

$$\|x(t,h)-\bar{x}(t)\|_{C^1} = O(h). \tag{5}$$

Here it is denoted that

$$\|z\|_{C^1} = \max_{[0,\omega]}|z| + \max_{[0,\omega]}|\dot{z}|.$$

Theorem 2. *If the conditions of Theorem 1 are satisfied and, in addition, the func-*

if f and τ are $n+1$ times continuously differentiable ($n \geqslant 1$), then the asymptotic (in C^1) formula is valid

$$x(t,h) = \sum_{k=0}^{n} h^k x_k(t) + O(h^{n+1}), \tag{6}$$

where $x_k(t)$ are ω-periodic solutions of ordinary differential equations, which are obtained as a result of substituting into equation (1), instead of x, the sum $\sum_{k=0}^{n} h^k x_k$, expanding the right-hand side in powers of h, and comparing the coefficients of like powers of h.

Thus, for example,

$$\dot{x}_0 = f(t,x_0,x_0,\dot{x}_0), \qquad x_0(t+\omega) \equiv x_0(t) \equiv \bar{x}(t),$$

$$\dot{x}_1 = (\bar{f}_x+\bar{f}_y)x_1 + \bar{f}_u\dot{x}_1 - \bar{f}_y\tau\dot{x}_0 - \bar{f}_u\tau\ddot{x}_0, \qquad x_1(t+\omega) \equiv x_1(t) \quad \text{and so on.}$$

Inequalities (3) and (4) ensure the existence and uniqueness of all $x_k(t)$.

Let us proceed to the proof of Theorem 1.

By the continuity of $\bar{f}_u$, the fulfillment of inequality (3) is possible in two cases: either $|\bar{f}_u| < 1$, or $|\bar{f}_u| > 1$. Consider the case $|\bar{f}_u| < 1$.

We shall prove the following lemma.

Lemma. Let the equation be given

$$\dot{x}(t) = b(t)x(t) + d(t)\dot{x}(t - h\tau(t)) + c(t), \tag{7}$$

where $b(t), d(t), c(t), \tau(t)$ are continuously differentiable ω-periodic functions and $h > 0$ is small; then, if

$$|d(t)| < 1, \qquad \int_0^\omega (1 - d(t))^{-1} b(t)\, dt \neq 0, \tag{8}$$

there exists, for sufficiently small h, a unique ω-periodic solution of equation (7).

Proof. Since $d(t)$ is continuous, there will be a constant number $a_0 < 1$ such that

$$|d(t)| \leqslant a_0 < 1. \tag{9}$$

Introduce the notation $[z] \equiv z(t - h\tau(t))$; then equation (7) can be rewritten as

$$\dot{x} = bx + d[\dot{x}] + c. \tag{7^1}$$

Substituting into the right-hand side of equation (7^1), in place of $\dot{x}$, its expression determined by the right-hand side of (7^1), we obtain

$$\dot{x} = bx + c + d[\dot{x}] = bx + c + d[bx + c] + d[d[\dot{x}]] = \ldots$$

$$\ldots = \sum_{k=0}^{m-1} d^k[bx + c]_k + d^m[\dot{x}]_m,$$

where the notation is

$$d^0[z]_0 \equiv z(t), \qquad d^k[z]_k \equiv d[d[\ldots d[z]\ldots]].$$

Thus, if $x(t, h)$ is an ω-periodic solution of equation (7^1), then $x(t, h)$ will also be a solution of the equation

$$\dot{x} = \sum_{k=0}^{m-1} d^k [bx + c]_k + d^m [\dot{x}]_m \tag{7^m}$$

for any $m \geqslant 1$.

We now fix m so that $a_0^m < \varepsilon$, where $\varepsilon > 0$ is a small number whose value will be specified below. We shall prove that, for this choice, equation (7^m) has a unique ω-periodic solution, provided h is small.

Under the conditions of the lemma there exists a unique ω-periodic solution $\bar{x}(t)$ of the degenerate equation

$$\dot{\bar{x}} = b\bar{x} + c + d\dot{\bar{x}},$$

which also satisfies the equation

$$\bar{x} = \sum_{k=0}^{m-1} d^k (b\bar{x} + c) + d^m \dot{\bar{x}}.$$

Denote $\Delta(t,h) = x(t,h) - \bar{x}(t)$. For the function $\Delta(t,h)$ we have the equation

$$\dot{\Delta} = \sum_{k=0}^{m-1} d^k b\Delta + d^m \dot{\Delta} + R(t,\Delta), \tag{10}$$

where

$$R(t,\Delta) = \sum_{k=0}^{m-1} d^k [b(\bar{x} + \Delta) + c]_k + d^m [\dot{x} + \dot{\Delta}]_m$$

$$- \sum_{k=0}^{m-1} d^k (b\bar{x} + c) - d^m \dot{\bar{x}} - \sum_{k=0}^{m-1} d^k b\Delta - d^m \dot{\Delta}.$$

We shall seek an ω-periodic solution of equation (10) by the method of successive approximations.

$$\dot{\Delta}_{n+1} = \sum_{k=0}^{m-1} d^k b\Delta_{n+1} + d^m \dot{\Delta}_{n+1} + R(t,\Delta_n); \qquad \Delta_{n+1}(t+\omega) = \Delta_n(t), \qquad \Delta_0 \equiv 0.$$

Inequalities (8) ensure the existence and uniqueness of the ω-periodic functions Δ_k. For the difference $z_n = \Delta_{n+1} - \Delta_n$ one can find

$$\|z_n\|_{C^1} < \frac{2\varepsilon}{1-\varepsilon}\,\|z_{n-1}\|_{C^1},$$

whence it is seen that, for small ε ($\varepsilon < 1/3$), the successive approximations converge uniformly to the periodic solution $\Delta(t,h)$.

Assuming the existence of two ω-periodic solutions Δ_1 and Δ_2, for their difference $z = \Delta_1 - \Delta_2$ we easily obtain

$$\|z\|_{C^1} \le \frac{2\varepsilon}{1-\varepsilon}\,\|z\|_{C^1}, \qquad z \equiv 0.$$

Thus, each of the equations (7^m), for $m \ge m_0(1/3)$, has a unique ω-periodic solution.

Let x be an ω-periodic solution of equation (7^m), and let $\tilde{x}$ be an ω-periodic solution of (7^{m+1}). Then it is clear that x will also be a solution of the equations (7^{2m}), (7^{3m}), ..., ($7^{(m+1)m}$). In turn, $\tilde{x}$ will satisfy the equations (7^{2m+2}), (7^{3m+3}), ..., ($7^{m(m+1)}$). Since equation (7^{m^2+m}) has a unique ω-periodic solution, it follows that $x \equiv \tilde{x}$. But then

$$x \equiv \sum_{k=0}^{m} d^k [bx + c]_k + d^{m+1}[\dot{x}]_{m+1} \equiv$$

$$\equiv bx + c + d\left[\sum_{k=0}^{m-1} d^k [bx + c]_k + d^m [\dot{x}]_m\right] \equiv bx + c + d[\dot{x}].$$

Thus, $x(t,h)$ is also an ω-periodic solution of equation (7). Uniqueness follows from uniqueness for equation (7^m). The lemma is proved.

It is clear from the proof of the lemma that

$$\|x(t,h) - \bar{x}(t)\|_{C^1} = O(h).$$

Proof of Theorem 1. Denote $\Delta(t,h) = x(t,h) - \bar{x}(t)$. We have

$$\dot{\Delta} = (\bar{f}_x + \bar{f}_y)\Delta + \bar{f}_u[\dot{\Delta}] + R(t,\Delta),$$

$$R(t,\Delta) = f(t, \bar{x} + \Delta, [\bar{x} + \Delta], [\dot{\bar{x}} + \dot{\Delta}]) - f(t, \bar{x}, \bar{x}, \dot{\bar{x}}) - (\bar{f}_x + \bar{f}_y)\Delta - \bar{f}_u[\dot{\Delta}].$$

Using the lemma, construct a sequence of periodic functions $\{\Delta_n(t,h)\}$

$$\dot{\Delta}_{n+1} = (\bar{f}_x + \bar{f}_y)\Delta_{n+1} + \bar{f}_u(\dot{\Delta}_{n+1}) + R(t, \Delta_n), \qquad \Delta_0 \equiv 0.$$

It is easy to obtain the estimate

$$\|\Delta_{n+1} - \Delta_n\|_{C^1} < ah\|\Delta_n - \Delta_{n-1}\|_{C^1},$$

from which follows the existence of an ω-periodic solution of equation (1) for small h. Uniqueness and estimate (5) are proved simply.

In the case $|\bar{f}_u| < 1$, Theorem 1 is proved.

If, however, $|\bar{f}_u| > 1$, then, solving equation (1) with respect to $[\dot{x}]$, we arrive at a case differing little from the one considered.

Theorem 1 is proved.

Theorem 2 can be proved by analogy with (5).

Remark 1. In the vector case, inequality (3) should be read as follows: the largest (smallest) modulus of an eigenvalue of the matrix $\bar{f}_u$ is less (greater) than one.

Inequality (4) will mean that the characteristic exponents of the system of ordinary differential equations with periodic coefficients

$$\dot{z} = (E - \bar{f}_u)^{-1}(\bar{f}_x + \bar{f}_y)z$$

are nonzero.

Remark 2. Analogous theorems also hold in the case of several deviations of the argument.

Remark 3. Similar results can also be obtained in the case where the degenerate system (2) has not one isolated ω-periodic solution, but a whole family of solutions.

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named after Patrice Lumumba

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Note: Figure translations are in progress. See original paper for figures.