ON THE THEORY OF CONVERGENCE OF APPROXIMATE SOLUTIONS OF AN OPERATOR EQUATION

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Abstract

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MATHEMATICS

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ON THE THEORY OF CONVERGENCE OF APPROXIMATE SOLUTIONS OF AN OPERATOR EQUATION

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Recently many works have appeared on the study of the convergence of approximate solutions of ill-posed problems ([1–5]). In the present note essentially the same questions are considered, but under different initial assumptions.

Consider in a Hilbert space H the linear equation

$$A\varphi = f \in H, \qquad A = A^*. \tag{1}$$

Let $e_1, e_2, \dots, e_n, \dots$ be an orthonormal system of eigenvectors of the operator A, with $Ae_i = 0$, $(f, e_i) = 0$ for $i = 1, 2, \dots, n$. We associate with equation (1) the equation

$$A_h\varphi_h = f_h \in H, \qquad h > 0, \tag{2}$$

where the linear operator A_h acts in the same Hilbert space H,

$$\|f - f_h\| = O(h^\alpha), \qquad \alpha > 0, \tag{3}$$

and on the solutions of equation (1) corresponding to $f \in H_1 \subset H$, the equalities

$$\|(A - A_h)\varphi\| = O(h^\beta), \qquad \beta > 0; \tag{4}$$

$$(A_h\varphi, e_i) = O(h^\varepsilon), \qquad i = 1, 2, \dots, n, \qquad \varepsilon > 0. \tag{5}$$

hold. Suppose, moreover, that

$$\|A_h^{-1}\| = O(h^{-\gamma}), \qquad \gamma \geq 0; \tag{6}$$

$$(f_h, e_i) = O(h^{\varepsilon_1}), \qquad i = 1, 2, \dots, n, \qquad \varepsilon_1 > 0. \tag{7}$$

We represent the solution φ_h of equation (2) in the form

$$\varphi_h = \varphi_0 + \varphi_1 + \varphi_2. \tag{8}$$

Here φ_0 is a solution of equation (1), and φ_1 is a solution of the equation

$$A\varphi_1 = -f_1, \tag{9}$$

where f_1 satisfies the equality

$$A_h\varphi_0 - f_h = f_1 + \tilde{f}_1, \tag{10}$$

$$\|\tilde{f}_1\| = O(h^{\mu}), \qquad \mu = \min(\varepsilon, \varepsilon_1).$$

From conditions (5), (7) it follows that the solution of equation (9) exists for $h^{-\nu} f_1 \in H_1$, and, by virtue of relations (3), (4),

$$\|f_1\| = O(h^{\nu}), \qquad \nu = \min(\alpha, \beta) \tag{11}$$

and, consequently, for $h^{-\nu} f_1 \in H_1$

$$\|\varphi_1\| = O(h^{\nu}). \tag{12}$$

Substituting (8) into equation (2), by virtue of the equalities (4), (6), (10), (11), (12), we obtain

$$\|A_h\varphi_2\| = \|f_h - A_h\varphi_0 - A_h\varphi_1\| = \| - f_1 - \tilde{f}_1 - A_h\varphi_1 + A\varphi_1 - A\varphi_1\| \leq$$

$$\leq \|\tilde{f}_1\| + \|(A - A_h)\varphi_1\| + \|(A\varphi_1 + f_1)\| = O\big(h^{\min(\mu,\, 2\nu)}\big), \tag{13}$$

$$\|\varphi_2\| = O\big(h^{\min(\mu,\, 2\nu) - \gamma}\big). \tag{14}$$

From (8), (12), (14) it follows that

$$\|\varphi_h - \varphi_0\| = O\big(h^{\min[\nu,\, \min(\mu,\, 2\nu) - \gamma]}\big). \tag{15}$$

Thus, we have proved the following

Theorem (convergence criterion). If $f, h^{-\nu} f_1 \in H_1$,

$$\lim_{h\to 0} \|\varphi_0 - \varphi_h\| = 0,$$

if

$$\mu = \min(\varepsilon, \varepsilon_1) > \min(\alpha, \beta) = \nu, \qquad \min(\mu, 2\nu) > \gamma.$$

Remark. It follows from relation (15) that the larger the instability exponent γ, the larger the approximation exponents μ and ν must be.

Example. As an example, let us consider the system of two linear algebraic equations

$$A\varphi = f = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{16}$$

It is obvious that the eigenvectors of the matrix of the system of equations (16) are

$$y_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad y_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{17}$$

We approximate equation (16) by the equation

$$A_h \varphi_h = f_h = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad A_h = \begin{pmatrix} 1 + h + h^2 & 1 \\ 1 & 1 + h \end{pmatrix}. \tag{18}$$

As a solution of the system (16) we take

$$\varphi_0 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}. \tag{19}$$

If H_1 is the set of vectors of the form

$$x = \begin{pmatrix} a \\ a \end{pmatrix}, \qquad |a| < a_0 = \text{const} < \infty, \tag{20}$$

then $f, h^{-1} f_1 \in H_1$ and $\mu = 2$, $\nu = 1$, $\gamma = 1$, $\min(\mu, 2\nu) = 2$. Consequently, according to the theorem, the solution of the system of equations (18) converges as $h \to 0$ to the solution (19) of equation (16).

The scheme of reasoning presented above is readily applied to the proof of convergence theorems also for evolution equations with unbounded operators.

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Note: Figure translations are in progress. See original paper for figures.