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WEIGHTED CUBATURE FORMULAS

MATHEMATICS

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Abstract

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MATHEMATICS

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WEIGHTED CUBATURE FORMULAS

(Presented by Academician S. L. Sobolev on 18 V 1967)

In the works of S. L. Sobolev ($^{1-5}$), cubature formulas were studied for a bounded domain of integration with constant weight and nodes at the points of regular lattices. In these works an algorithm was indicated for constructing sequences of cubature formulas that are asymptotically optimal in the class $L_2^{(m)}(E)$. For the error functionals of optimal cubature formulas, asymptotically unimprovable estimates of norms in $L_2^{(m)*}(E_n)$ were obtained. In the present note the main results of ($^{1-5}$) are generalized to the case in which the weight function is integrable in the square.

We introduce the following notation: $L_2^{(m)}(E_n)$ is the space of classes of functions possessing, in the n-dimensional space E_n, generalized derivatives integrable in the square up to order m (we shall assume that $2m > n$) and with norm

$$\|f\|_{L_2^{(m)}(E_n)} = \left\{ \int_{E_n} \sum_{|\alpha|=m} (D^\alpha f)^2 \, dx \right\}^{1/2} \tag{1}$$

(for an integral vector α, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$, $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}$); Ω is a domain in E_n with piecewise smooth boundary Γ; H is a square matrix of order $n \times n$; Ω_0 is a fundamental domain for the matrix H, i.e., such that the characteristic function $\mathcal{E}_{\Omega_0}$ of the domain Ω_0 satisfies, for all x, the identity

$$\sum_\gamma \mathcal{E}_{\Omega_0}(x - H\gamma) \equiv 1; \tag{2}$$

$\Omega_{h,\gamma}$ are domains such that $\mathcal{E}_{\Omega_{h,\gamma}}(x) \equiv \mathcal{E}_{\Omega_0}(x/h - H\gamma)$; h is a numerical parameter taking positive values.

Definition 1. A set of interpolation operators $\{I^h\}$ is called a **family of uniformly distributed operators** if all the operators of $\{I^h\}$ are defined on $L_2^{(m)}(E_n)$, i.e., they assign to each function f, $f \in L_2^{(m)}(E_n)$, functions $I^h f$, and are representable in the form of sums of operators I_γ^h,

$$I^h = \sum_{\gamma} I^h_\gamma, \tag{3}$$

possessing the following properties:

a)

$$(I^h_\gamma f)(x) = \sum_{|\gamma'| \le L} g^h_{\gamma,\gamma+\gamma'}(x) f(Hh(\gamma + \gamma')), \tag{4}$$

where $g^h_{\gamma,\gamma+\gamma'}(x)$ are functions equal to zero outside $\Omega_{h,\gamma}$; L is a constant independent of h and γ;

b)

$$\left| g^h_{\gamma,\gamma+\gamma'}(x) \right| \le M, \tag{5}$$

M is a constant independent of h, γ, and γ';

c)

$$I^h_\gamma \left(x_1^{\alpha_1}, x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right) = \mathcal{E}_{\Omega_{h,\gamma}} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad \text{for } |\alpha| \le m. \tag{6}$$

Theorem 1. For every family of uniformly distributed operators $\{I^h\}$ there exists a number A such that, for any weight function g square-integrable in Ω, for the norms in $L_2^{(m)*}(E_n)$ of the functionals of the errors of the cubature formulas

$$\int_\Omega gf\, dx \cong \int_\Omega g I^h f\, dx \tag{7}$$

l^h_g, the estimates

$$\|l^h_g\|_{L_2^{(m)*}(E_n)} \le A \left[\int_\Omega |g|^2\, dx \right]^{1/2} h^m = A\|g\|_{L_2(\Omega)} h^m \tag{8}$$

hold.

We outline the proof of Theorem 1. Define the functionals $l^{h,\gamma}_g$ by the equalities

$$l^{h,\gamma}_g(f) = \int_{\Omega \cap \Omega_{h,\gamma}} g[f - I^h f]\, dx. \tag{9}$$

Analogously to the proof of equality (24) from (^{1}), the validity of the formula

$$\|l_g^h\|^2_{L_2^{(m)*}(E_n)} = \sum_{\gamma_1}\sum_{\gamma_2}\left[G(x) * l_g^{h,\gamma_1}(x) * l_g^{h,\gamma_2}(-x)\right]_0 \tag{10}$$

is shown.

Here $G(x)$ in (10) is the fundamental solution of the polyharmonic equation

$$\Delta^m u = 0. \tag{11}$$

Lemma 1. The estimate

$$\left|G(x) * l_g^{h,\gamma_1}(x) * l_g^{h,\gamma_2}(-x)\right| \leq Kh^{2m}\frac{\|g\|_{L_2(\Omega_{h,\gamma_1})}\|g\|_{L_2(\Omega_{h,\gamma_2})}}{[1+|H(\gamma_1-\gamma_2)|^2]^{n/2+1}}. \tag{12}$$

holds.

The proof of Lemma 1 is analogous to the proof of Lemma 2 from $(^1)$. Substituting (12) into (10) and applying Theorem 275 from $(^6)$, p. 239, we obtain the assertion of Theorem 1.

Definition 2. A set of interpolation operators $\{I^h\}$ is called a **family of interpolation operators with a regular boundary layer** if $\{I^h\}$ is a uniformly distributed family and there exists a constant T characterized by the following property: if the integer vectors γ_1, γ_2 are such that the distances from the points $Hh\gamma_1, Hh\gamma_2$ to Γ are greater than Th, then the functions $g^h_{\gamma_1,\gamma_1+\gamma}(x)$, $g^h_{\gamma_2,\gamma_2+\gamma}(x)$, for all γ, $|\gamma| \leq L$, satisfy the identities

$$g^h_{\gamma_1,\gamma_1+\gamma}(x) \equiv g^h_{\gamma_2,\gamma_2+\gamma}(x - hH\gamma_1 + hH\gamma_2). \tag{13}$$

From the definition of functionals with a regular boundary layer it follows that

Lemma 2. If the family of interpolation operators $\{I^h\}$ is uniformly distributed, then the set of functionals $\{l_\xi^h\}$ is a set of functionals with a regular boundary layer satisfying the conditions of the main theorem from $(^2)$.

Theorem 2. If $\{I^h\}$ is a family of interpolation operators with a regular boundary layer, and g is a weight function of class $L_2(\Omega)$, then for the norms in $L_2^{(m)*}(E_n)$ of the functionals l_g^h, as $h \to 0$, the estimate

$$\|l_g^h\|_{L_2^{(m)*}(E_n)} = (2\pi)^{-m}\|g\|_{L_2(\Omega)}h^m\sqrt{\xi(H^{-1}, 2m)} + o(h^m) \tag{14}$$

holds.

Theorem 2 is first proved for the case of a piecewise constant weight, and then, with the help of Theorem 1, is extended to the general case. At the first stage

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of the proof, Lemma 2, the main theorem from (2), and representation (10) for the norm of l_g^h in $L_2^{(m)*}(E_n)$ are used essentially.

Definition 3. A cubature formula is said to be **optimal** in $L_2^{(m)*}(E_n)$ if the norm of its error functional $l_{g,0}^h$ is minimal in $L_2^{(m)*}(E_n)$ among the norms of all functionals of the form:

$$l = g\mathcal{E}_\Omega - \sum_{\substack{\gamma \\ |hH\gamma| \in \Omega}} c_\gamma \delta(x - hH\gamma), \qquad l\left(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\right) = 0 \quad \text{for } |\alpha| \leq m-1, \tag{15}$$

where c_γ are constant coefficients.

Theorem 3. If $\{I^h\}$ is a family of interpolation operators with a regular boundary layer, and g is a function in $L_2(\Omega)$, then as $h \to 0$ the estimate

$$\left\|l_g^h\right\|_{L_2^{(m)*}(E_n)} - \left\|l_{g,0}^h\right\|_{L_2^{(m)*}(E_n)} = o(h^m) \tag{16}$$

holds.

The proof of Theorem 3 is based on approximating the weight function g by piecewise-constant functions. The proof essentially uses Theorem 1, Theorems 5 and 6 from (3), and the following lemma, whose validity follows from the Hilbert nature of the spaces $L_2^{(m)}(E_n)$ and $L_2^{(m)*}(E_n)$.

Lemma 3. For weight functions g_1 and g_2 integrable in Ω, the optimal cubature formulas have error functionals $l_{g_1,0}^h$ and $l_{g_2,0}^h$ satisfying the equality

$$l_{g_1,0}^h + l_{g_2,0}^h = l_{g_1+g_2,0}^h. \tag{17}$$

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Note: Figure translations are in progress. See original paper for figures.

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