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Abstract

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MATHEMATICS

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ON THE GROUP CLASSIFICATION OF SECOND-ORDER DIFFERENTIAL EQUATIONS

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The paper considers the problem of group classification of second-order differential equations with $n \geqslant 3$ independent variables, based on works ([1],[2]). Determining equations are obtained for infinitesimal transformations admitted by one type of quasilinear second-order equations of physical interest. An invariant Laplace equation in a Riemannian space is introduced.

1. In physics (see ([3]), where a detailed bibliography is also given), equations of the form

$$F[u] \equiv g^{ij}(x)\partial^2 u/\partial x^i \partial x^j + b^i(x)\partial u/\partial x^i + \psi(x, u) = 0, \tag{1}$$

$$(g^{ij} = g^{ji};\ i, j = 1, \dots, n;\ \det \|g^{ij}\| \neq 0),$$

are considered when $n = 4$, and the terms with derivatives are simply the wave operator. Here we shall consider the arbitrary case of equation (1) for $n \geqslant 3$.

Introduce a Riemannian space V_n with metric tensor $g_{ij}(x)$, defined by the equalities $g_{ik}g^{kj} = \delta_i^j$ $(i, j = 1, \dots, n)$, and use the usual notation of tensor analysis: Γ_{ij}^k are the Christoffel symbols, R_{ij} is the Ricci tensor, $R = R_i^i$ is the scalar curvature of the space V_n. Indices after a comma will denote covariant differentiation. We shall consider equations up to equivalence, calling the following equivalence transformations:

$$x'^i = x'^i(x) \quad (i = 1, \dots, n), \tag{2a}$$

$$F'[u] = e^{-f} F[ue^f], \tag{2b}$$

$$F'[u] = \Phi(x) F[u]. \tag{2c}$$

The problem considered is that of finding continuous transformation groups admitted by equations (1) in the sense of S. Lie. Omitting the standard calculations, we give here only the results.

The infinitesimal operator of a continuous one-parameter local Lie group admitted by equation (1) has the form

$$X = \xi^i(x)\frac{\partial}{\partial x^i} + [\sigma(x)u + \tau(x)]\frac{\partial}{\partial u} \tag{3}$$

and is determined from the equations

$$\begin{gathered} \xi_{i,j} + \xi_{j,i} = \mu(x)g_{ij}, \\ 2\sigma_{,i} = \frac{2-n}{2}\mu_{,i} - (a_j\xi^j)_{,i} - K_{ij}\xi^j, \\ (K_{il}\xi^l)_{,j} = (K_{jl}\xi^l)_{,i}, \\ u(\xi^{kH}_{,k} + \mu H) + X(\psi) + (\mu - \sigma)\psi + g^{ik}\partial^2\tau/\partial x^i\partial x^k + b^i\partial\tau/\partial x^i = 0, \end{gathered} \tag{4}$$

where

$$a^i = b^i + g^{jk}\Gamma^i_{jk}; \quad K_{ij} = a_{i,j} - a_{j,i}; \quad H = -\frac{1}{2}\left(a^i{}_{,i} + \frac{1}{2}a^{ia}_i + \frac{n-2}{2(n-1)}R\right).$$

From the first line of equations (4) it follows that, as in the case of linear equations (1), the quantities ξ^i yield a subgroup of the group of conformal transformations of the space V_n. It is known that the maximal number of parameters of the group of conformal transformations of the space V_n $(n \geqslant 3)$ does not exceed $(n+1)(n+2)/2$, and this number is attained only for conformally flat spaces ([4]). For groups of motions the maximal number of parameters is $n(n+1)/2$, and it is attained only for spaces of constant curvature ([5]). Using these facts, after some computations we arrive at the following theorem.

Theorem 1. *Equations of the form (1) that admit a group of motions of the space V_n of maximal order $n(n+1)/2$ are equivalent, with respect to the transformations $(2^a), (2^b), (2^c)$, to one of the following equations:*

$$\Delta u + u^{(n+2)/(n-2)}\varphi\left(u(1+r^2)^{(n-2)/2}\right) = 0 \quad \left(r^2 = \sum_{i=1}^n e_i(x^i)^2\right) \tag{5}$$

in the case of nonzero constant curvature of the space V_n, or

$$\Delta u + \varphi(u) = 0 \tag{6}$$

in the case of zero curvature, where φ is an arbitrary function. If, however, equation (1) admits a group of conformal transformations of the space V_n of maximal order $(n+1)(n+2)/2$, then it is equivalent to the equation

$$\Delta u + au^{(n+2)/(n-2)} = 0, \tag{7}$$

where a is an arbitrary constant.

Here by Δ we have denoted the operator

$$\Delta = \sum_{i=1}^{n} e_i \frac{\partial^2}{(\partial x^i)^2}, \qquad e_i = \pm 1,$$

and the signature of the operator coincides with the signature of g^{ij}. In what follows we shall call this operator the Laplace operator, without regard to the signature.

As a consequence of Theorem 1 we obtain that the standard form of a linear second-order equation admitting a continuous group of maximal order $(n+1)(n+2)/2$ (not counting the transformations $u' = au + \varphi(x)$, where $a = \text{const}$, and $\varphi(x)$ is an arbitrary solution of the equation under consideration) is the Laplace equation

$$\Delta u = 0. \tag{8}$$

2. We see that, in considering linear second-order equations, the Laplace equation (8) is characteristic for conformally flat Riemannian spaces. We now introduce an invariant Laplace equation (in particular, the wave equation for the corresponding signature) in a Riemannian space, possessing an analogous property with respect to spaces conformal to one another. It is customary to regard as the Laplace equation in a Riemannian space the equation

$$\Delta_2 c \equiv g^{ij} u_{,ij} = 0, \tag{9}$$

obtained by the simple replacement of ordinary partial derivatives by covariant ones. The basis for this is the external similarity between equation (9) and the ordinary Laplace equation (8), as well as the fact that (9) passes into (8) when the space V_n is flat. However, there is an essential difference between equations (8) and (9). Namely, if (8) is invariant with respect to the group of conformal transformations of flat space, then (9), generally speaking, is invariant only with respect to the group of motions, and not of conformal transformations, of the corresponding space V_n (i.e., of the space whose metric tensor components are ···).

are the coefficients g^{ij} at the second derivatives). This is easy to verify, for example, when equation (9) is written in a space of constant nonzero curvature, by solving the determining equations (4). It seems reasonable to us, proceeding from the properties of the ordinary Laplace equation (8), to require that the Laplace equation in V_n satisfy the following conditions:

1°. Linearity and homogeneity.
2°. General covariance.
3°. Invariance with respect to the group of conformal transformations of the space V_n.

Theorem 2. *In every Riemannian space V_n the equation*

$$\Delta u \equiv g^{ij} u_{,ij} + \frac{n-2}{4(n-1)} R u = 0 \tag{10}$$

satisfies conditions $1°, 2°, 3°$. For a conformally flat space such an equation is unique up to an equivalence transformation (2^b).

On the basis of this theorem the following is proposed.

Definition. Equation (10) is called the **invariant Laplace equation in the Riemannian space** V_n, and the operator Δ appearing in (10) is called the invariant Laplace operator in V_n.

It follows from this definition that, for spaces V_n with zero scalar curvature R, the invariant Laplace equation takes the form (9). This occurs, for example, for Einstein spaces ($R_{ij} = 0,\ i, j = 1, \ldots, n$).

The invariant Laplace equation has the following remarkable property.

Theorem 3. *Let the Riemannian spaces V_n and $\overline{V}_n$ be conformal to each other, and let the components of their metric tensors be respectively*

$$g_{ij}(x), \quad \overline{g}_{ij}(x) = e^{2\theta(x)} g_{ij}(x).$$

Then the invariant Laplace equations in these spaces are equivalent to each other, and the equivalence transformation has the form

$$\overline{\Delta} u = e^{-\frac{n+2}{2}\theta} \Delta \left(u e^{\frac{n-2}{2}\theta} \right), \tag{11}$$

where Δ and $\overline{\Delta}$ are the invariant Laplace operators defined in (10) for the spaces V_n and $\overline{V}_n$, respectively.

Equations (5) and (6) can be combined into one and written in the form (6), if by the operator Δ one understands the invariant Laplace operator in a space of constant curvature. Therefore the standard forms found in ([1]) of second-order linear equations admitting motion groups of maximal order take the form

$$\Delta u - \lambda u = 0 \qquad (\lambda = \text{const}) \tag{12}$$

with the invariant Laplace operator Δ. It follows from ([1]) that, in a space of constant curvature, equation (12) is the only (up to equivalence transformations) generally covariant equation admitting a motion group of maximal order. Equation (12), written for an arbitrary Riemannian space V_n, will be called the **invariant Klein-Gordon equation in V_n.**

As an example, let us consider the application of the definition introduced here of the invariant Laplace equation in presenting the theory of redshift on the basis of the hypothesis that space-time is a Friedman-Lobachevsky space ([6]). Let the square of the interval be pro-

of the Friedmann-Lobachevsky space has the form

$$ds^2 = H^2(x)\left(dt^2 - dx^2 - dy^2 - dz^2\right). \tag{13}$$

We shall not specify the form of the function $H(x)$. Suppose that the propagation of light in V_4 obeys the wave equation in this space. If we assume that the wave equation in V_n has the form (10), then from the invariant Laplace equation (or wave equation)

$$\Delta\psi = 0, \tag{14}$$

written in Friedmann-Lobachevsky space, it follows, by virtue of formula (11), that

$$\Delta\psi = \frac{1}{H^3}\Box(\psi H), \tag{15}$$

where $\Box$ is the ordinary wave operator in flat space. Making the transformation

$$\psi^* = \psi H,$$

we obtain from (14) and (15) the equation

$$\Box\psi^* = 0. \tag{16}$$

If, however, equation (9) is taken as the wave equation in a Riemannian space, then instead of (16) one obtains the equation ([6]), §95

$$H\Box\psi^* - \psi^*\Box H = 0, \tag{17}$$

whence, using the special form of the function $H(x)$ and neglecting the term $\Box H$, we obtain, approximately, equation (15). The physical consequences of the transition from the wave equation in Friedmann–Lobachevsky space to the ordinary wave equation (16) are set forth in ([6]).

We have introduced here the invariant Laplace equation (10) formally, proceeding only from group-theoretic considerations. Equally formal, of course, is the assumption made in the example given that the propagation of light in the space V_4 is described by the invariant Laplace equation (10). The question of the physical content of the invariant Laplace equation in V_n (in particular, the question of the correspondence between equation (10) and Maxwell' s equations in V_4) remains open.

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Note: Figure translations are in progress. See original paper for figures.