# ON CLASSIFYING SPACES FOR PRINCIPAL BUNDLES WITH HAUSDORFF BASES

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## Abstract

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**MATHEMATICS**

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# ON CLASSIFYING SPACES FOR PRINCIPAL BUNDLES WITH HAUSDORFF BASES

In our recent work ([1]) (see also ([2,3])) we defined an analogue of the universal bundle for principal $G$-bundles with Hausdorff bases, where $G$ is a closed Lie subgroup of the group $GL(n, \mathbf{R})$. The present note contains a generalization of this construction, making it possible to construct a universal $G$-bundle for any topological group $G$. This construction contains as special cases both the construction from article ([1]) (with a slight modification) and an analogue of the well-known construction of Milnor ([6]). With its help one can extend the results of article ([1]) to the case of an arbitrary connected Lie group and establish, for any paracompact group $G$, the coincidence of the group $H^q_{alg}(G; V)$ of characteristic classes of principal $G$-bundles (with Hausdorff bases) with coefficients in a topological $G$-module $V$ and the group $H^q_c(G; V)$ of "continuous cohomology of Eilenberg-Mac Lane" ([4]).

1. Let $G$ be a topological group. By a principal $G$-bundle we mean a triple consisting of a topological space $E$, on which the group $G$ acts on the right without fixed points (i.e., if for some $g \in G$, $y \in E$ the equality $yg = y$ holds, then $g = e$), of the quotient space $X$, and of the projection $p : E \to X$. The principal $G$-bundle $\xi = (E, p, X)$ is called locally trivial in the classical sense (or simply locally trivial) if every point $x \in X$ has a neighborhood $U$ such that the full inverse image $p^{-1}(U) \subset E$, as a $G$-space, is isomorphic to the product $U \times G$.

By an elementary $G$-object $\mathbf{T}$ we shall mean a triple $\mathbf{T} = (T, A, \mu)$, where $T$ is a Hausdorff space with a distinguished point $*$; $A$ is a topological semigroup with identity, containing the group $G$; $\mu : T \times A \to T$ is a continuous action of the semigroup $A$ on $T$ such that:

1°. Every transformation $a \in A$ of the space $T$ leaves the point $*$ fixed.

2°. Distinct elements of the semigroup $A$ define distinct transformations of the space $T$.

3°. There exists a homotopy $\varphi_t : T \to T$ such that $\varphi_t \in A$ for all $t$, $\varphi_0$ is the identity transformation, and $\varphi_1(z) = *$ for all $z \in T$.

4°. There exist a neighborhood $W$ of the group $G$ in the semigroup $A$ and a mapping $\lambda : W \to G$ such that: (a) every element of $W$ is invertible in the semigroup $A$; (b) $ag \in W$ for all $a \in W$, $g \in G$; (c) $\lambda(ag) = \lambda(a)g$ for all $a \in A$, $g \in G$; (d) $\lambda(g) = g$ for every $g \in G$.

Let us note that the group $G$ acts both on the space $T$ and on the semigroup $A$ (by right multiplications); moreover, the invertible elements of the semigroup $A$ form a $G$-invariant subset on which the group $A$ acts without fixed points.

The principal examples of elementary objects for us will be the following two.

**Example 1.** *$G$ is a closed Lie subgroup of the group $GL(n, \mathbf{R})$, $T = \mathbf{R}^n$, and $A$ is the semigroup of all linear mappings of $\mathbf{R}^n$ into itself.*

**Example 2.** *$G$ is any topological group, $T$ is the cone $CG$ over $G$ (i.e., the set of pairs $(g, t)$, where $g \in G$, $t$ is a nonnegative number, ···*

whereby the pairs $(g', 0)$ and $(g'', 0)$ are identified for any $g', g'' \in G$. The semigroup $A$, as a topological space, coincides with $T$; multiplication in $A$ (and the action of $A$ on $T$) is defined by the formula

$$(g_1, t_1)(g_2, t_2) = (g_1 g_2, t_1, t_2).$$

The embedding $G \subset A$ sends an element $g \in G$ to the point $(g, 1) \in A$. The element $\varphi_t$ is chosen to be equal to $(e, 1 - t)$.

Fix a group $G$ and an elementary $G$-object $T = (T, A, \mu)$. With every principal $G$-bundle $\xi = (E, p, X)$ one can associate the bundle $\xi_T(E_T, p_T, X)$ with fiber $T$, by putting $E_T = (E \times T)/G$ (the action of the group $G$ on $E \times T$ is defined coordinatewise). To each element $y \in E$ there corresponds a map $\eta_y : T \to E_T$ (the composition $T = y \times T \subset E \times T \to E_T$), homeomorphically mapping $T$ onto the fiber of the bundle $\xi_T$ over the point $p(y)$. Obviously, $\eta_{yg} = \eta_y g$ for all $y \in E$, $g \in G$, and distinct points $y_1, y_2 \in E$ correspond to distinct maps $\eta_{y_1}, \eta_{y_2}$. It is also clear that the map $\eta_y$ depends continuously on the point $y$.

**Definition.** A principal $G$-bundle $\xi = (E, p, X)$ is called **locally $T$-trivial** if for every point $x \in X$ there exists a continuous map $\pi_x : E_T \to T$ such that for all $y \in E$ the map $\pi_x \eta_y : T \to T$ belongs to the semigroup $A$, and for $py = x$ is its invertible element.

**Remark.** The definition would not change if we required that, for $py = x$, the element $\pi_x \eta_y \in A$ belong to $G \subset A$: to satisfy this condition it suffices to take, instead of the map $\pi_x$, the map $(\pi_x \eta_y)^{-1} \pi_x$, where $py = x$.

Obviously, the bundle induced by a locally $T$-trivial bundle $\xi = (E, p, X)$ under any map $f : X' \to X$ is also locally $T$-trivial.

**Proposition 1.** *Every locally $T$-trivial bundle is locally trivial (in the classical sense).*

**Proof.** Let $x$ be an arbitrary point of the base of the locally $T$-trivial bundle $\xi = (E, p, X)$. As was noted, we may assume that the map $\pi_x \eta_y$ is an element of the group $G$ for $py = x$. The set of those points $y \in E$ for which $\pi_x \eta_y \in W$ is open in $E$. Since $\pi_{yg} = \eta_y g$, this set is invariant under the action of the group $G$ on $E$, i.e. it is the full inverse image of some open set $U \subset X$ under the projection $p$. The map

$$p^{-1}(U) \to U \times G,$$

sending an element $y \in p^{-1}(U)$ to the element $(p(y), \lambda(\pi_x \eta_y)) \in U \times G$, is compatible with the action of the group $G$. Consequently, the $G$-spaces $p^{-1}(U)$ and $U \times G$ are isomorphic.

**Proposition 2.** *If the base $X$ of a locally trivial bundle $\xi$ is completely regular, then $\xi$ is locally $T$-trivial.*

**Proof.** Let $x \in X$ be a point; $\sigma : U \to E$ a section of the surface of the bundle $\xi$ over a neighborhood $U \ni x$; $h$ a continuous real-valued function on $X$, taking values from 0 to 1, equal to 0 at the point $x$ and to 1 outside $U$. Put

$$\pi_x(z) = \varphi_{h(z)} \eta^{-1}_{\sigma(p_T(z))}(z)$$

for $p_T(z) \in U$, and

$$\pi_x(z) = *$$

for the remaining $z$.

**2.** We proceed to describe the construction of the universal bundle. For a topological space $X$ and a set $I$, denote by $X^I$ the topological space whose points are families $\{x_i\}$ of points of the space $X$, indexed by the elements of the set $I$. A basis of open sets of the space $X^I$ is formed by the sets

$$\Gamma(i_1, \dots, i_k; U_1, \dots, U_k),$$

where $i_1, \dots, i_k \in I$, $U_1, \dots, U_k$ are open sets in $X$,

$$\Gamma(i_1, \dots, i_k; U_1, \dots, U_k) = \{\{x_i\} \mid x_{i_1} \in U_1, \dots, x_{i_k} \in U_k\}.$$

Denote by $\mathcal{E}^I_G$ the subspace of the space $A^I$ consisting of those families $\{a_i\}$ in which at least one element $a_i$ is invertible in the semigroup $A$. We define the action of the group $G$ on $\mathcal{E}^I_G$ by the formula $\{a_i\}g = \{a_i g\}$. The quotient space $\mathcal{E}^I_G/G$ will be denoted by $\mathcal{B}^I_G$, and the projection $\mathcal{E}^I_G \to \mathcal{B}^I_G$ by $p^I_G$, or, for short, by $p$.

**Proposition 3.** *The space $\mathcal{B}^I_G$ is Hausdorff.*

**Proof** is obvious.

**Proposition 4.** The bundle $(\mathcal{E}^I_G, p^I_G, \mathcal{S}^I_G)$ is locally $T$-trivial.

**Proof.** The points of the space $\mathcal{E}^I_G$ may be identified with those embeddings $T \to T^I$ whose composition with the projection $\rho_i : T^I \to T$, given by the formula $\rho_i(\{z_i\}) = z_i$ for any $i \in I$, belongs to the semigroup $A$. The points of the space $(\mathcal{E}^I_G)_T$ may be identified with pairs $(\varphi, z)$, where $\varphi$ is such an embedding and $z \in \varphi(T)$; moreover $(\varphi', z) = (\varphi'', z)$ if $\varphi' = \varphi'' g$, $g \in G$. The projection $(\mathcal{E}^I_G)_T \to \mathcal{S}^I_G$ consists in discarding the second element of the pair. Denote by

$$\pi_i : (\mathcal{E}^I_G)_T \to T \quad (i \in I)$$

the mapping given by the formula $\pi_i(\varphi, z) = \rho_i(z)$. The requirements in the definition of local $T$-triviality will be satisfied by the mapping $\pi_x : (\mathcal{E}^I_G)_T \to T$, which is defined for each point $x \in \mathcal{S}^I_G$ as the mapping coinciding with the mapping $\pi_i$, where $i \in I$ is such an element that there exists a point $y = \{a_i\} \in \mathcal{E}^I_G$ for which $p(y) = x$ and the element $a_i$ is invertible in $A$.

**Theorem 1.** For every locally $T$-trivial bundle $\xi = (E, p, X)$ such that the cardinality of a basis of open sets of the space $X$ does not exceed the cardinality of the set $I$, there exists a continuous mapping $\varphi : X \to \mathcal{S}^I_G$ such that the bundle $\xi$ is equivalent to the bundle induced from $(\mathcal{E}^I_G, p, \mathcal{S}^I_G)$ by means of $\varphi$.

**Proof.** It is enough to construct a continuous mapping of the space $E$ into the space $\mathcal{E}^I_G$, equivariant with respect to the action of the group $G$. For each point $y \in E$ there is defined a mapping $\eta_y : T \to E_T$ such that for any $x \in X$ with $p(y) = x$, the mapping $\pi_x \eta_y : T \to T$ is an invertible element of the semigroup $A$. From item 4° of the definition of an elementary $G$-object it follows that the elements $\pi_x \eta_y$ will be invertible in the semigroup $A$ also for those $y \in E$ for which $p(y) \in U_x$, where $U_x$ is some neighborhood of $x$. The neighborhoods $U_x$ form coverings of the space, into which, by assumption, one may inscribe a covering whose cardinality does not exceed the cardinality of the set $I$. Therefore from all the mappings $\pi_x$ one may select a collection of mappings $\{\pi_i\}$, indexed by the elements of the set $i$, such that for each point $y \in E$ at least one of the mappings $\pi_i \eta_y$ will be invertible in the semigroup $A$. It remains to define the mapping $\Phi : E \to \mathcal{E}^I_G$ by putting $\Phi(y) = \{\pi_i \eta_y\}$.

Thus, the bundle $\xi^I_G = (\mathcal{E}^I_G, p, \mathcal{S}^I_G)$ is universal for locally $T$-trivial principal $G$-bundles whose bases have a basis of neighborhoods whose cardinality does not exceed the cardinality of the set $I$.

Let us dwell in more detail on two basic examples of elementary $G$-objects. If $G$ is a Lie group, $T = \mathbb{R}^n$, and $A$ is the semigroup of all linear mappings, then $\mathcal{E}^I_G$ is the space of all linear embeddings

$$\mathbb{R}^n \to (\mathbb{R}^n)^I$$

such that the composition with one of the projections

$$(\mathbb{R}^n)^I \to \mathbb{R}^n$$

is nondegenerate. Recall that in the paper ([1]), as the space $\mathcal{E}^I_G$ we considered the space of all linear embeddings

$$\mathbb{R}^n \to (\mathbb{R}^n)^I.$$

Thus the universal bundle obtained from the general construction set forth is a subbundle of the universal bundle from the paper ([1]), and the base of the former is an everywhere dense open set in the base of the latter.

If $T = CG$, then the space $\mathcal{E}^I_G$ is obtained from $(CG)^I$ by deleting the single point $\{a_i\}$, where $a_i = *$ (the vertex of the cone) for all $I$. Let us note that if $T$ is a set of $n$ elements, then the space $\mathcal{E}^I_G$ is homeomorphic to the direct product of the $(n-1)$-fold join

$$G * \ldots * G$$

of the group $G$ with the line $\mathbb{R}$, where the action of the group does not change the coordinate of the point on $\mathbb{R}$. Recall that Milnor' s classical construction of the universal bundle for the group $G$ consists in defining the action of the group on the space

$$E_n = \underbrace{G * \ldots * G}_{n},$$

and then as⋯

as the universal bundle for the group $G$ one takes $(E_G, p, B_G)$, where $E_G = \varinjlim E_n$ and $B_G = E_G/G$. Thus, in the case $T = CG$ our construction gives an analogue of Milnor' s universal bundle.

**3. Definition.** One says that a $q$-dimensional **characteristic class** of locally $T$-trivial principal $G$-bundles with coefficients in the topological $G$-module $V$ is given if to each locally $T$-trivial principal $G$-bundle $\xi = (E, p, X)$ there is assigned an element $a(\xi) \in H^q(X; V)$, where $V$ is the sheaf of germs of sections of the bundle with fiber $V$ induced by the bundle $\xi$, and, moreover, for any map $\varphi : \xi' = (E', p', X') \to \xi'' = (E'', p'', X'')$ the equality $\bar{\varphi}^* a(\xi'') = a(\xi')$ holds. Here $\bar{\varphi} : X' \to X''$ is the map of bases corresponding to the map $\varphi$.

The characteristic classes form an abelian group, which is denoted by $H^q_{alg}(G; V)$.

**Definition.** The group $H^q_c(G; V)$ of continuous cohomology of Eilenberg-Mac Lane of the group $G$ with coefficients in the topological $G$-module $V$ is the $q$-th cohomology group of the complex

$$F_0 \xrightarrow{\partial_0} F_1 \xrightarrow{\partial_1} \cdots,$$

where $F_q$ is the group of continuous maps of the product $G \times \cdots \times G$ (with $q$ factors) into $V$, and

$$\partial_q f(g_1, \ldots, g_{q+1}) = g_1 f(g_2, \ldots, g_{q+1})$$

$$\sum_i (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{q+1}) + (-1)^{q+1} f(g_1, \ldots, g_q).$$

**Theorem 2.** *For every paracompact group $G$ the equality*

$$H^q_{alg}(G;V) = H^q_c(G;V)$$

*holds. (In particular, the group $H^q_{alg}(G;V)$ does not depend on $T$.)*

**Proof** of this theorem is based on the fact that if the characteristic classes $a'$ and $a''$ coincide for all bundles $\xi^I_G$, then they are equal to one another. The following lemma plays the decisive role.

**Lemma.** *For every continuous real-valued function on the space $\mathcal{E}^I_G$ there is a subset $I' \subset I$ such that the difference $I \setminus I'$ is at most countable, and the restriction $f|_{\mathcal{E}^{I'}_G}$ is constant.*

The proof is the same as the proof of the analogous lemma in (1) (see Lemma 2.4).

Recall that in the paper (1) it was proved for closed subgroups $G$ of the group $GL(n;\mathbf{R})$ that the group $H^q_{alg}(G;\mathbf{R})$ is isomorphic to the $q$-th cohomology group of the complex of $G$-invariant differential forms on the space $G/\widehat{G}$, where $\widehat{G} \subset G$ is a maximal compact subgroup. The isomorphism of this group with the group $H^q_c(G;V)$ was established by Hochschild and Mostow (5).

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*Note: Figure translations are in progress. See original paper for figures.*