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STATISTICAL DISTRIBUTION FOR A TURBULENT FLOW

HYDROMECHANICS

1968

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Abstract

Full Text

UDC 532.517.4

HYDROMECHANICS

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STATISTICAL DISTRIBUTION FOR A TURBULENT FLOW

(Presented by Academician M. D. Millionshchikov, 15 XII 1967)

1. The central problem in the theory of turbulent flows (of a liquid, gas, or plasma) is to find a stationary probability distribution analogous to the Gibbs distribution in statistical physics. Of course, the existence of such a distribution does not yet solve all practical problems. For utilitarian purposes it will be necessary to calculate multiple integrals of this distribution with one weight or another, which, as is known, even in the case of the Gibbs distribution is a rather difficult matter. Nevertheless, finding the general statistical distribution for turbulent flows is of fundamental importance.

The main difficulty in solving the indicated problem is the presence of energy dissipation in a turbulent flow. Energy is not conserved, but passes through the system; therefore the Gibbs ideas are not directly applicable here. However, there is nevertheless something invariant with respect to different realizations of turbulent motion—namely, the equations of dynamics themselves. It will be shown below how this strange idea, under a certain assumption of the unique solvability of the equations of dynamics, makes it possible to construct a probability distribution for a turbulent flow.

For definiteness we shall consider an incompressible viscous fluid. The possibility of generalizing the distribution obtained below to the case of a compressible gas and plasma is sufficiently clear.

The use of the Navier–Stokes equations in describing turbulent motions requires a certain caution. It is known that stationary solutions at large Reynolds numbers become unstable. Nonstationary solutions may also lose stability. Under real conditions there always exist perturbations (thermal fluctuations, vibration of the walls, etc.) which excite unstable degrees of freedom and ultimately lead to the turbulent regime. However, the equations “do not know” about the existence of these perturbations. Therefore, especially when solving a statistically stationary problem, where the initial velocity field does not appear as a source of perturbations, something must be added to the equations in order not to find oneself in the position of people describing, with the help of equations, unstable solutions that have no physical meaning. In the present work this difficulty is

overcome by adding to the energy sources (such as the mean pressure gradient in a pipe) weak random forces which do not make a substantial contribution to the energy balance of the flow, but at the same time possess a sufficiently broad spectrum to excite all possible degrees of freedom.*

The general solution of the problem of the evolution of a turbulent velocity field in the presence of random forces is expressed through the special type of continual integral introduced in (2) (this is the general solution obtained auto-

* Another route, outlined in (1), consists in seeking a distribution satisfying additional conditions that filter out undesirable solutions.

by Kármán in 1961, was included in the monograph (3). However, the volume of computations needed to extract concrete results from the general solution (integration over six functions, each of which depends on four arguments) exceeds the capabilities of existing computers; moreover, some mathematical questions relating to this remain as yet unclear. Below another path is proposed, which in principle already now makes it possible to proceed to numerical experiments.

2. For what follows it is important to reduce the integro-differential equations of motion (the integrals arise when the pressure is eliminated) to a system of algebraic equations. To this end, let us expand the fields of velocities and forces in some complete system of functions and, in addition, impose on the flow the condition of periodicity in time with a sufficiently large period T . Let us denote the set of the corresponding Fourier coefficients by the symbols v_α and f_α , where $\alpha = 1, 2, \dots$ is a discrete parameter. From the Navier–Stokes equations with a random force on the right-hand side we have:

$$\sum_{\beta=1}^{\infty} g_{\alpha\beta} v_\beta + \sum_{\beta,\gamma=1}^{\infty} h_{\alpha\beta\gamma} v_\beta v_\gamma = f_\alpha \quad (\alpha = 1, 2, \dots), \quad (1)$$

where $g_{\alpha\beta}$ and $h_{\alpha\beta\gamma} = h_{\alpha\gamma\beta}$ are numerical coefficients whose explicit form depends on the geometry of the flow and on the chosen system of functions. Let us note that f_α includes the mean value $\langle f_\alpha \rangle$, i.e., the sources of energy.

We shall proceed further by Galerkin's method. Denote by \tilde{v}_α an approximate solution of system (1). Setting $\tilde{v}_\alpha = 0$ for $\alpha > n$ (i.e., neglecting sufficiently high harmonics, which in principle should not be very significant owing to the action of viscosity), we obtain a finite-dimensional approximation of equations (1):

$$\sum_{\beta=1}^n g_{\alpha\beta} \tilde{v}_\beta + \sum_{\beta,\gamma=1}^n h_{\alpha\beta\gamma} \tilde{v}_\beta \tilde{v}_\gamma = f_\alpha \quad (\alpha = 1, \dots, n). \quad (2)$$

Using the energy-balance equation, which gives an a priori estimate for the solution, and Brouwer's theorem on the existence of a fixed point under a

mapping of a sphere into itself ⁽⁴⁾, one can prove that system (2) has a solution for any finite f ⁽⁵⁾. However, uniqueness can be proved only for sufficiently small f . Nevertheless, let us now assume that uniqueness holds for “almost all f ” (in the sense of probability measure). We equate the increments of probabilities

$$W_T^{(n)}(\tilde{v}) \prod_{\alpha=1}^n d\tilde{v}_\alpha = F_T^{(n)}(f) \prod_{\alpha=1}^n df_\alpha. \quad (3)$$

Here $W_T^{(n)}$ and $F_T^{(n)}$ are the corresponding n -dimensional distribution densities, depending parametrically on the period T . Introduce the Jacobian of the transformation (2):

$$J_T^{(n)}(\tilde{v}) = \text{Det} \left\{ g_{\alpha\beta} + 2 \sum_{\gamma=1}^n h_{\alpha\beta\gamma} \tilde{v}_\gamma \right\}. \quad (4)$$

From (3), taking (2) into account, we finally obtain:

$$W_T^{(n)}(\tilde{v}) = F_T^{(n)} \left(\sum g\tilde{v} + \sum h\tilde{v}\tilde{v} \right) |J_T^{(n)}(\tilde{v})|. \quad (5)$$

The normalization condition

$$\int_{-\infty}^{\infty} W_T^{(n)}(\tilde{v}) \prod_{\alpha=1}^n d\tilde{v}_\alpha = 1, \quad (6)$$

where for $W_T^{(n)}(\tilde{v})$ expression (5) must be used, may serve

criterion of uniqueness almost everywhere. Indeed, denoting by $S_T^{(n)}(f)$ the number of solutions corresponding to a given f , we have $S_T^{(n)}(f) \geq 1$, and in a neighborhood of zero $S_T^{(n)}(f) = 1$. If all solutions are assigned the same statistical weight, then, replacing in the right-hand side of (5) $F_T^{(n)}$ by $F_T^{(n)}/S_T^{(n)}$, we obtain an expression for the probability density satisfying the normalization condition. Without introducing the indicated additional factor, the integral of the right-hand side of (5) will be greater than unity if uniqueness is violated on a set of finite measure. Let us note that uniqueness certainly holds if the Jacobian (4) does not vanish. For the given problem it is sufficient that the condition

$$\lim_{T, n \rightarrow \infty} \int_{-\infty}^{\infty} W_T^{(n)}(\tilde{v}) \prod_{\alpha=1}^n d\tilde{v}_\alpha = 1. \quad (7)$$

be satisfied.

Thus, under the assumption of uniqueness of the solution of the dynamical equation almost everywhere, we have obtained explicitly the solution of the statistical problem. Various spectral characteristics are expressed through the n -fold integral of (5) with one weight or another.

As was already indicated above, the random forces must possess a broad spectrum and a sufficiently small intensity of fluctuations. The condition

$$\sigma^2 = \sum_{\alpha=1}^n \langle (f'_\alpha)^2 \rangle \ll \sum_{\alpha=1}^n \langle f_\alpha \rangle^2; \quad f'_\alpha = f_\alpha - \langle f_\alpha \rangle. \quad (8)$$

must certainly be satisfied. If the spectral tensor of the forces

$$\mathcal{F}_{\alpha\beta} = \langle f'_\alpha f'_\beta \rangle \quad (9)$$

is made to tend to zero, then, obviously,

$$F_T^{(n)}(f) \rightarrow \prod_{\alpha=1}^n \delta(f_\alpha - \langle f_\alpha \rangle). \quad (10)$$

In this limiting case, computation of the integral of (5) with any weight is in fact equivalent to the numerical solution of system (2) and the substitution of this solution into the weight function. The presence of unstable solutions makes this procedure mathematically incorrect and simply nonunique. The introduction of small (but finite) random forces is equivalent to a smearing of the distribution density (10). In this case the integral of (5) already becomes a quite unambiguous integral with respect to a measure, which can be evaluated, for example, by the Monte Carlo method. Random forces thus produce a kind of regularization of the problem. It is natural to suppose that, for sufficiently small values of $\mathcal{F}_{\alpha\beta}$, the results of calculations by means of (5) for such principal flow characteristics as the mean-velocity profile, the friction stress, the energy spectrum, and so on, will not depend on the magnitudes $\mathcal{F}_{\alpha\beta}^*$ or on the form of the distribution density $F_T^{(n)}(f)$. This distribution may be chosen to be the simplest and most convenient for calculations, for example Gaussian:

$$F_T^{(n)}(f) = (2\pi)^{-n/2} [\text{Det}\{\mathcal{F}_{\alpha\beta}\}]^{-1/2} \exp \left\{ -\frac{1}{2} \sum_{\alpha,\beta=1}^n \mathcal{F}_{\alpha\beta}^{-1} f'_\alpha f'_\beta \right\}, \quad (11)$$

where $\mathcal{F}_{\alpha\beta}^{-1}$ is the matrix inverse to $\mathcal{F}_{\alpha\beta}$.

* As $\mathcal{F}_{\alpha\beta}$ is decreased, generally speaking, the accuracy of the calculations increases, but the volume of computations also increases (one must increase the number of points so as not to miss the sharper maximum in the subintegral expression). The optimal choice of these quantities can for the time being be

made only by numerical experiments. The author hopes to return in the future to the mathematical theorem needed here.

3. Random forces can also be used in another formulation of the problem, namely, in finding a statistical distribution for the local structure of a turbulent flow that would satisfy Kolmogorov's similarity hypothesis⁶. In this case one considers a homogeneous and isotropic flow whose energy is maintained by the work of random forces (the mean value of the forces is now equal to zero).

In order to pass to a finite-dimensional approximation, let us consider a flow that is periodic in space and in time, with corresponding periods L and T . We represent the velocity and force fields in the form of space-time Fourier series and denote the set of corresponding coefficients by v_α and f_α . Thus we again arrive at equations of the form (1), but with simpler coefficients (in particular, the matrix $g_{\alpha\beta}$ is now diagonal). The further consideration is entirely analogous, and the final result has the form (5). We note that, in this model formulation of the problem, random forces were used in work⁷; however, in the formula analogous to (5), the second factor—the Jacobian of the transformation—was omitted there. The random forces (which are no longer small) may be chosen the same as in^{7,8}, i.e., Gaussian, with a spectrum independent of frequency and concentrated in the region of small wave numbers. The principal parameter characterizing these forces is the mean value of the energy influx $\langle \varepsilon \rangle$, which corresponds to the similarity hypothesis⁶.

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Received
13 XI 1967

CITED LITERATURE

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