A NEW CLASS OF EXACT SOLUTIONS OF THE BIHARMONIC PROBLEM FOR A SEMI-STRIP

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Abstract

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A NEW CLASS OF EXACT SOLUTIONS OF THE BIHARMONIC PROBLEM FOR A SEMI-STRIP

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Many practically important problems of the theory of elasticity and thermoelasticity reduce to the investigation of the stress state of a semi-strip. From the extensive literature on this question we shall confine ourselves to citing papers (^{1}-^{5}), which contain a detailed bibliography. The class of exact solutions of the problem is in fact limited to Papkovich functions (^{6}), describing a semi-strip with free longitudinal edges and an end loaded by a self-equilibrated system of normal and tangential forces. Continuing the solution through the end of the semi-strip in order to use known methods for investigating the stress state of an infinite strip and to satisfy the conditions at the end of the semi-strip by expanding the solution in a series in Papkovich functions constitutes the usual basis of various approaches to solving the problem of loading a semi-strip.

We shall show that continuing the solution not through the end, but through the longitudinal edges of the semi-strip, opens up new ways of solving the problem and makes it possible, in particular, to construct new complete systems of exact solutions. By periodically continuing the solution for the semi-strip $y \geq 0,\ 0 \leq x \leq 2\pi$ to the half-plane $y \geq 0$, we obtain a half-plane with additional sources of stress located at points with abscissas that are multiples of 2π. These sources will be concentrated body forces equal to the sum of the forces acting on both edges of the semi-strip at points with the corresponding abscissa, and dislocations (incompatibility of strains) with density determined by the difference of displacements on the edges of the semi-strip. In other words, right-hand sides will appear in the equations of equilibrium of stresses and in the equations of compatibility of strains, replacing the boundary conditions on the edges of the semi-strip. Conversely, any representation of the solution in the form of a trigonometric series (for example, in the form of a cosine-sine series (^{7})) is equivalent to continuing the solution to the half-plane with some choice of additional stress sources replacing the accounting of boundary conditions.*

For definiteness let us consider the case of a semi-strip with free end and longitudinal edges, free of tangential forces, but symmetrically loaded by some system of normal forces that produces a known distribution of normal displacements

$$u(0,y) = -u(2\pi,y) = f(y).$$

The stress function

$$\psi(x,y) = \psi(2\pi - x, y),$$

describing the stress state of such a semi-strip, satisfies inside the semi-strip the homogeneous biharmonic equation

$$\Delta^2\psi = 0 \tag{1}$$

and the boundary conditions

$$\psi(x,0) = \partial\psi(x,0)/\partial y = 0; \tag{2}$$

* Analogous assertions can be made for a rectangle. Here double trigonometric series continue the solution to the plane with certain sources of stress.

$$\partial\psi(0,y)/\partial y = \partial\psi(2\pi,y)/\partial y = 0; \tag{3}$$

$$\frac{\partial^2}{\partial y^2}\int_0^{2\pi}\psi(x,y)\,dx = E[u(2\pi,y) - u(0,y)] = -2Ef(y), \tag{4}$$

where E is Young' s modulus. Condition (2) describes the absence of normal and tangential loads on the end of the semistrip; condition (3), the absence of tangential loads on the edges; condition (4) is obtained by integrating Hooke' s law for the normal strains e_{xx}, taking into account the self-equilibration condition for the stresses σ_{yy} that follows from (3). As the limiting condition as $y \to \infty$, it is sufficient to require boundedness of the stresses.

The continuation of the solution to the half-plane $y \geq 0$ satisfies the nonhomogeneous biharmonic equation

$$\Delta^2\psi = 2E\frac{d^2 f}{dy^2}\sum_{m=-\infty}^{\infty}\delta(x - 2\pi m) \tag{5}$$

with the above-mentioned limiting conditions and the boundary conditions (2), which are satisfied along the entire boundary of the half-plane. Thus, the problem of loading of the semistrip has been reduced to the problem of edge dislocations periodically distributed in the half-plane with density

$$\beta(x,y) = 2\delta(x - 2\pi m)\,df/dy. \tag{6}$$

Using the known solution ([8]) for an edge dislocation with Burgers vector $b_x = b$, located at the point $(0,t)$ of the half-plane,

$$\psi_0 = \frac{Eb}{2\pi}\left\{-\frac{1}{2}(y-t)\ln\frac{x^2+(y-t)^2}{x^2+(y+t)^2} + 2yt\frac{y+t}{x^2+(y+t)^2}\right\}, \tag{7}$$

we construct the solution for an infinite row of edge dislocations with $b_x = 1$, located at the points $(2\pi m, t)$:

$$\Psi(x,y,t) = \frac{E}{2\pi b}\sum_{m=-\infty}^{\infty}\psi_0(x-2\pi m, y, t) =$$

$$= \frac{E}{2\pi}\left\{-\frac{(y-t)}{2}\ln\frac{\operatorname{ch}(y-t)-\cos x}{\operatorname{ch}(y+t)+\cos x} + yt\,\frac{\operatorname{sh}(y+t)}{\operatorname{ch}(y+t)-\cos x}\right\}. \tag{8}$$

Here, in evaluating the sums

$$\sum_{-\infty}^{\infty}\ln\sqrt{(x-2\pi m)^2+(y\pm t)^2}, \qquad \sum_{-\infty}^{\infty}\frac{y+t}{(x-2\pi m)^2+(y+t)^2},$$

the known representation of the sine as an infinite product has been used. Expression (8) satisfies the equation

$$\Delta^2\Psi = 2E\sum_{m=-\infty}^{\infty}\delta(x-2\pi m)\delta'(y-t) \tag{9}$$

and serves as the Green' s function for equation (5), whose solution we can now write in the form

$$\psi(x,y) = \int_0^{\infty}\Psi(x,y,t)\frac{df(t)}{dt}\,dt. \tag{10}$$

The stresses and displacements corresponding to the prescribed normal component of displacement of the edges of the semistrip are found from the stress function in the usual way. In particular, the normal displacement of the end of the semistrip is given by the equation

$$v(x) = \int_0^{\infty}V(x,t)\frac{df(t)}{dt}\,dt, \tag{11}$$

where

$$V(x,t) = \frac{1}{E}\frac{\partial^2}{\partial x^2}\int_0^{\infty}\Psi(x,y,t)\,dy = \frac{t}{\pi}\left(\frac{\operatorname{sh} t}{\operatorname{ch} t-\cos x}-1\right). \tag{12}$$

It should be noted that the use of formula (11) requires some caution, since, generally speaking, one cannot interchange the order of the operations of summing the contribution of an infinite series of sources and integrating with respect to the ordinate over infinite limits. The indicated difference, however, reduces only to the appearance of a constant term in the normal displacement of the edge of the half-strip for functions f that do not satisfy the condition $f(0) = f(\infty)$.

Thus, the functions $\Psi(x, y, t)$ form a complete system of functions describing the stressed state of a half-strip with a free end and edges free of tangential stresses and acquiring the stepwise form

$$u(0, y) = -u(2\pi, y) = \begin{cases} 1, & y \le t, \\ 0, & y > t, \end{cases} \tag{13}$$

under the action of normal loads

$$\sigma_{xx}(0, y) = \sigma_{xx}(2\pi, y) = \partial^2\Psi(0, y, t)/\partial y^2. \tag{14}$$

It is clear that, in an analogous manner, other solutions may be continued to the half-plane for the loading of the half-strip, including solutions for another symmetry of loads on the edges and solutions for a loaded end. Summation of the contribution of stress sources in all cases gives a complete system of functions describing the loaded state of the half-strip, but, depending on the manner in which the boundary conditions are specified, it may be necessary to pass from the specification of sources to the actual boundary conditions, requiring the solution of the corresponding integral equation. In our case, for example, the functions $\Psi(x, y, t)$ immediately give a solution of the problem for prescribed normal displacements, while for prescribed normal stresses $\sigma_{xx}(0, y)$ it is necessary to find from the integral equation

$$\sigma_{xx}(0, y) = \int_0^\infty \Phi(t)\, \frac{\partial^2\Psi(0, y, t)}{\partial y^2}\, dt \tag{15}$$

the function $\Phi(t)$, with the aid of which the required stress function is written directly as

$$\psi(x, y) = \int_0^\infty \Phi(t)\Psi(x, y, t)\, dt. \tag{16}$$

The continuation of the solution to the half-plane can be interpreted as the determination of a solution for the half-strip in the form of a trigonometric series

$$\psi(x, y) = \sum_{n=-\infty}^{\infty} f_n \cos nx. \tag{17}$$

Since the original equation (1) is not satisfied at the points $x = 2\pi m$, the functions $\psi_n = f_n(y) \cos nx$ need not satisfy the homogeneous biharmonic equation. Indeed, expanding both sides of (5) in a Fourier series and using the well-known expression for the sum of δ-functions

$$\sum_{-\infty}^{\infty} \delta(x - 2\pi m) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \cos nx, \tag{18}$$

we obtain

$$\Delta^2 \psi_n = -\frac{E}{\pi} \frac{d^2 f}{dy^2} \cos nx, \tag{19}$$

whence

$$f_n^{\mathrm{IV}} - 2n^2 f_n'' + n^4 f_n = \frac{E}{\pi} \frac{d^2 f}{dy^2},$$

i.e., the Fourier components of the sought stress function correspond to the Fourier components of the dislocation distribution (6) with nonzero dislocation density (and strain incompatibility) inside the half-strip.

For equation (9), expansion of the solution in a trigonometric series gives

$$f_n^{\mathrm{IV}} - 2n^2 f_n'' + n^4 f_n = \frac{E}{\pi} \delta'(y - t), \tag{20}$$

where the functions f_n satisfy the boundary conditions $f_n(0) = f_n'(0) = 0$ at the end of the half-strip and the natural limiting conditions at infinity. Hence, for $n \neq 0$,

$$f_n = \frac{E}{\pi n} \left\{ [t + (2nt - 1)y] e^{-n(y+t)} + (y - t) e^{-n(y-t)} \right\}, \tag{21}$$

and for the amplitude of the zero harmonic we obtain $f_0 = \dfrac{E}{2\pi} y^2$ for $y \leq t$, and $f_0 = \dfrac{E}{2\pi} t(2y - t)$ for $y > t$ (in the case of a strip of length L, in order to satisfy the equilibrium conditions, one should add to f_0 the term $\dfrac{E}{2\pi} y^2 \left[\left(1 - \dfrac{t}{L}\right)^3 - 1 \right]$).

Summation of the functions $\psi_n = f_n \cos nx$ frees the interior of the strip from dislocations and again gives the solution (8).

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Note: Figure translations are in progress. See original paper for figures.

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