AN A PRIORI ESTIMATE OF THE PRINCIPAL RADII OF CURVATURE OF A CLOSED CONVEX HYPERSURFACE DEPENDING ON ITS GAUSSIAN CURVATURE

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Abstract

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Let F be a regular hypersurface in $(n+1)$-dimensional Euclidean space, and let P be an arbitrary point on this hypersurface. Take the point P as the origin of a rectangular Cartesian coordinate system $x_1, x_2, \dots, x_n, z$, and the tangent hyperplane at the point P as the coordinate hyperplane $z = 0$. In these coordinates a neighborhood of the point P of the hypersurface can be given by the equation

$$z = f(x_1, x_2, \dots, x_n).$$

The eigenvalues k_i of the second differential $\frac{1}{2}d^2f$ at the point P are called the principal curvatures, and the reciprocal quantities R_i are the principal radii of curvature. The total, or Gaussian, curvature of a hypersurface is the product of the principal curvatures.

The purpose of the present note is to establish an a priori estimate for the principal radii of curvature of a closed convex hypersurface in terms of its Gaussian curvature, given as a function of the unit vector of the outward normal. For a two-dimensional surface in three-dimensional space such an estimate was obtained by Miranda (^{1}) and by the author (^{2}).

Theorem. *Let F be a closed strictly convex hypersurface in $(n+1)$-dimensional Euclidean space satisfying the equation*

$$R_1 R_2 \dots R_n = \varphi(\nu) > 0, \tag{1}$$

where $R_1, R_2, \dots, R_n$ are the principal radii of curvature of the hypersurface at the point with outward normal ν.

Then for the principal radii of curvature R_i the estimate holds

$$R_i \leqslant \max_{X,\gamma} \varphi^{1/n} \left(n - 1 + \frac{1}{2} \left(\frac{\varphi'}{\varphi} \right)^2 - \frac{\varphi''}{\varphi} \right)^{1-1/n},$$

where differentiation is performed along the arc of a great circle γ *issuing from the point* X *on the spherical image* Ω *of the hypersurface, and the maximum is taken over all* X *and* γ*.*

Proof. Consider the function $w_\gamma(X)$, defined on the unit sphere Ω by the equality

$$w_\gamma(X) = \overline{H} + \overline{H}''_\gamma,$$

where $\overline{H}$ is the support function of the hypersurface F, considered on the unit sphere Ω, and differentiation is performed along the arc of the great circle γ issuing from the point X. The function $w_\gamma(X)$ is the radius of normal curvature of the projecting cylinder and lies between the largest and smallest radii of normal curvature of the hypersurface at the point with outward normal $\nu(X)$. If the great circle γ has a principal direction, this function coincides with the corresponding principal radius of curvature. It follows from this that an estimate for the function $w_\gamma(X)$ is at the same time an estimate for the principal radii of curvature.

Let the function $w_\gamma(X)$ attain its maximum at a certain point X_0 for some large circle γ_0 passing through this point. Introduce a Cartesian coordinate system in space, taking the direction $\nu(X_0)$ as the direction of the z-axis, and direct the axes $x_1, x_2, \dots, x_n$ parallel to the principal directions of the surface F at the point with exterior normal $\nu(X_0)$, with the axis x_1 directed parallel to the tangent to γ_0.

The coordinate net $x_1, x_2, \dots, x_n$ on the plane $z = 1$ is projected from the origin (the center of the sphere Ω) onto a certain curvilinear coordinate net on the sphere Ω. Define on the hemisphere $\Omega^+ (z > 0)$ the function

$$w(x_1, x_2, \dots, x_n) = w_{\gamma_1}(x_1, \dots, x_n),$$

where γ_1 is a large circle along which $x_i = \text{const}$ $(i > 1)$. The function $w(x_1, \dots, x_n)$ has the same maximum as $w_\gamma(X)$, and this maximum is attained at the point $x_1 = x_2 = \dots = x_n = 0$.

Denote by $h(x_1, x_2, \dots, x_n)$ the values of the support function of the hypersurface F on the hyperplane $z = 1$. The function w is expressed in terms of the function h, and for it one obtains the value

$$w = \frac{\partial^2 h}{\partial x_1^2} \frac{(1 + x_1^2 + \dots + x_n^2)^{3/2}}{1 + x_2^2 + \dots + x_n^2}.$$

The product of the principal radii of curvature is also expressed in terms of the function h:

$$R_1 R_2 \cdots R_n = (1 + x_1^2 + \cdots + x_n^2)^{n/2+1} \left|\partial^2 h / \partial x_i \partial x_j\right|.$$

On the right-hand side of the equality there stands the determinant of the second derivatives of the function h. We shall agree to denote differentiation with respect to x_i by the corresponding indices. Then

$$w = h_{11} \frac{(1 + x_1^2 + \cdots + x_n^2)^{3/2}}{1 + x_2^2 + \cdots + x_n^2}, \tag{2}$$

$$(1 + x_1^2 + \cdots + x_n^2)^{n/2+1} \left\|h_{ij}\right\| = \varphi. \tag{3}$$

With our choice of coordinate system, at the point X_0, i.e. for $x_1 = x_2 = \cdots = x_n = 0$, we shall have

$$h_{ii} = R_i, \qquad h_{ij} = 0 \quad \text{for } i \neq j.$$

Differentiating equality (2) at the point X_0, where w attains a maximum, we obtain

$$w_i = h_{11i} = 0, \tag{4}$$

$$w_{11} = (h_{11})_{11} + 3R_1 \leq 0, \qquad w_{ii} = (h_{11})_{ii} + R_1 \leq 0, \quad i \neq 1. \tag{5}$$

Differentiating equality (3) at the point X_0 with respect to x_1, we shall have

$$\sum_i h_{11} \cdots (h_{ii})_1 \cdots h_{nn} = \varphi$$

or

$$\varphi \sum_i \frac{(h_{ii})_1}{h_{ii}} = \varphi_1. \tag{6}$$

Differentiating equation (3) at the point X_0 twice with respect to x_1, we obtain

$$(n+2)\varphi + \varphi \sum_i \frac{(h_{ii})_{11}}{h_{ii}} + \varphi \sum_{i \neq j} \frac{(h_{ii})_1}{h_{ii}} \frac{(h_{jj})_1}{h_{jj}} = \varphi_{11}. \tag{7}$$

Taking into account equality (6), we conclude that

$$\sum_{i\neq j} \frac{(h_{ii})_1}{h_{ii}} \frac{(h_{jj})_1}{h_{jj}} \leq \frac{1}{2}\left(\frac{\varphi_1}{\varphi}\right)^2. \tag{8}$$

By the property of the maximum of the function w at the point X_0, from inequality (5) we obtain

$$\frac{(h_{11})_{11}}{h_{11}} \leq -3, \qquad \frac{(h_{ii})_{11}}{h_{ii}} \leq -\frac{R_1}{R_i} \quad \text{for } i \neq 1. \tag{9}$$

From equality (7) and inequalities (8) and (9) we conclude that

$$(n-1)\varphi - \varphi R_1 \sum_{i>1} \frac{1}{R_i} + \frac{1}{2}\frac{\varphi_1^2}{\varphi} \geq \varphi_{11}.$$

Hence, noting that

$$\sum_{i>1} \frac{1}{R_i} \geq (n-1)\left(\frac{1}{R_2} \cdots \frac{1}{R_n}\right)^{1/(n-1)} = (n-1)\left(\frac{R_1}{\varphi}\right)^{1/(n-1)},$$

we finally obtain the required estimate

$$R_1 \leq \varphi^{1/n}\left(n-1+\frac{1}{2}\left(\frac{\varphi_1}{\varphi}\right)^2 - \frac{\varphi_{11}}{\varphi}\right)^{1-1/n}.$$

Differentiation with respect to x_1 at the point X_0 can be replaced by differentiation along the arc of the circle γ_0.

The theorem is proved.

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CITED LITERATURE

[1] C. Miranda, Rend. Semin. mat. Roma, 4 (1939).
[2] A. V. Pogorelov, Mat. Sb., 31 (73), No. 1 (1952).

Note: Figure translations are in progress. See original paper for figures.

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