# FUNCTIONAL FORMULATION OF BOGOLYUBOV' S AXIOMS DEFINING A CAUSAL \(S\)-MATRIX THEORY

MATHEMATICAL PHYSICS

1968

SovietRxiv

## Abstract

## Full Text

UDC 530.145

*MATHEMATICAL PHYSICS*

L. Sh. KHODJAEV

# FUNCTIONAL FORMULATION OF BOGOLYUBOV'S AXIOMS DEFINING A CAUSAL $S$-MATRIX THEORY

*(Presented by Academician N. N. Bogolyubov on 16 II 1968)*

In the present work we shall give a functional formulation of the axioms of the $S$-matrix approach in quantum field theory, developed by N. N. Bogolyubov, B. V. Medvedev, M. K. Polivanov ([1]), and others. Bogolyubov's axioms encompass not only those properties possessed by the operators of free fields, but also certain general and local properties reflecting physical principles and the functional structures of quantities. For simplicity we restrict ourselves to a single self-interacting real scalar field.

**A. Axioms characterizing general properties.**

**Axiom I. Equipped Hilbert space.**
To an asymptotic state of the system, representing a collection of some number of particles infinitely remote from one another, there correspond vectors of an equipped Hilbert space $D_S^{\text{ex}} \subset H_{L_2}^{\text{ex}} \subset D_S'^{\text{ex}}$,

$$D_S^{\text{in}} \subset H_{L_2}^{\text{in}} \subset D_S'^{\text{in}} = D_S^{\text{out}} \subset H_{L_2}^{\text{out}} \subset D_S'^{\text{out}}, \quad \text{where ex} = \{\text{in}, \text{out}\}.$$

Here, by

$$D_S^{\text{ex}} = \bigoplus_{n=0}^{\infty} S(\overline{\Omega}_n^+)$$

we denote the nuclear Fock space ([2,3]), consisting of vectors

$$\Phi = \{\Phi^{(0)}, \Phi^{(1)}(p_1), \dots, \Phi^{(n)}(p_1, \dots, p_n), \dots\} = \{\Phi^{(+)}(p)_n\}_0^{\infty}, \tag{1}$$

whose components $\Phi^{(n)}(p_1, \dots, p_n)$, $n = 1, 2, \dots$, belong to the space $S(\overline{\Omega}_n^+)$ of basic functions, symmetric and

$$\operatorname{supp} \Phi^{(n)}(p_1, \dots, p_n) \subset \overline{\Omega}_n^+(p_1, \dots, p_n) = \bigoplus_{j=1}^{n} \overline{\Omega}_\mu^+(p_j),$$

where

$$\overline{\Omega}_\mu^+(p_j) = \{p_j \in \widetilde{R}^4 : \ (p_j^0)^2 - (\mathbf{p}_j)^2 = \mu^2, \ p_j^0 > 0\}, \quad j = 1, 2, \dots, n.$$

Moreover, $\Phi^{(n)}(p_1, \dots, p_n) = 0$ for $n > N$, where $N$ is some positive number. $D_S^{\text{ex}}$ is a linear topological space, and the scalar product in $D_S^{\text{ex}}$ is defined according to

$$(\Phi, \Psi) = \Phi^0 \Psi^0 + \sum_{n=1}^{\infty} \int \cdots \int \left( \prod_{j=1}^{n} d^4 p_j \, \theta(p_j^0) \, \delta(p_j^2 - \mu^2) \right) \overline{\Phi}^{(n)}(p)_n \, \Psi^{(n)}(p)_n. \tag{2}$$

Let $H_{L_2}^{\text{ex}}$ be the Hilbert space obtained from $D_S^{\text{ex}}$ by completing it with respect to the nondegenerate scalar product (2). Then, by

by definition, $D_S^{\text{ex}} \subset H_{L_2}^{\text{ex}} \subset D_S'^{\text{ex}}$ is a rigged Hilbert space ([4,5]) of vectors of, respectively, regular ordinary and generalized states, where $D_S'^{\text{ex}} = \bigoplus_{n=0}^{\infty} S'(\bar{\Omega}_n^+)$ is the space of vector-valued generalized functions denoted by $|\rho\rangle = \{|\rho^{(n)}(p_1, \dots, p_n)\rangle\}_0^\infty$, where

$$|\rho^{(n)}(p_1, \dots, p_n)\rangle = \left( \prod_{j=1}^{n} \theta(p_j^0) \delta(p_j^2 - \mu^2) \right) a^+(p_1) \dots a^+(p_n) |0\rangle \in S'(\bar{\Omega}_n^+),$$

and

$$|\rho(\Phi)\rangle = \sum_{n=0}^{\infty} |\rho^{(n)}(\Phi^{(n)})\rangle, \tag{3}$$

where

$$|\rho^{(n)}(\Phi^{(n)})\rangle = \int \dots \int \left( \prod_{j=1}^{n} \theta(p_j^0) \delta(p_j^2 - \mu^2) \right) \times$$

$$\times \Phi^{(n)}(p_1, \dots, p_n) a^+(p_1) \dots a^+(p_n) |0\rangle.$$

**Axiom II. Poincaré invariance.** To each transformation $(a, \Lambda)$ of the proper Poincaré group $P_+^\uparrow$ there corresponds a unitary representation $U(a, \Lambda)$ in the rigged Hilbert space $D_S \subset H_{L_2} \subset D'_S$, where the 4-vector $a$ denotes a translation, and $\Lambda$ a transformation of the proper Lorentz group $L^{\uparrow}_{+}$.

**Axiom III. Spectral condition and completeness.** There exists a system of generalized proper state amplitudes of the 4-momentum operator $\hat{P}^\nu$, $\nu = 0, 1, 2, 3$ (the generator of space-time translations), corresponding to eigenvalues $p^\nu \in \bar{V}^+$, where $\bar{V}^+ = \{p \in \bar{R}^4 : p^2 = (p^0)^2 - (\mathbf{p})^2,\ p^0 > 0\}$ and $p^\nu = 0$.

IIIa. If, in the generalized state $|\rho^{(1)}(p)\rangle \in {S'}^{(1)}(\bar{\Omega}^+_\mu)$, the 4-momentum operator $\hat{P}^\nu$ has values $p^\nu \in \bar{\Omega}^+_\mu$, i.e.

$$\hat{P}^\nu|\rho^{(1)}(\Phi^{(1)})\rangle = |\rho^{(1)}(\hat{P}^\nu\Phi^{(1)})\rangle, \quad \text{where } \hat{P}^\nu\Phi^{(1)}(p) = p^\nu\Phi^{(1)}(p), \tag{4}$$

for any $\Phi^{(1)}(p) \in S(\bar{\Omega}^+_\mu)$, then

$$U(a,\Lambda)|\rho^{(1)}(\Phi^{(1)})\rangle = |\rho^{(1)}(\Phi^{(1)}_{(a,\Lambda)})\rangle, \quad \text{where } \Phi^{(1)}_{(a,\Lambda)}(p) = e^{iap}\Phi^{(1)}(\Lambda^{-1}p) \tag{5}$$

for any $\Phi^{(1)}(p) \in S(\bar{\Omega}^+_\mu)$.

IIIb. There exists a nondegenerate state corresponding to the eigenvalue $p^\nu = 0$, for which

$$U(a,\Lambda)|0\rangle = |0\rangle \tag{6}$$

is the vacuum state, and moreover

$$|0\rangle \in D_S. \tag{7}$$

IIIc. By virtue of the completeness of the system of generalized proper state amplitudes of the 4-momentum, $\hat{P}^\nu$, the completeness condition holds:

$$\langle\rho|A(f)B(g)|\eta\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\alpha_1\dots\alpha_n}\langle\rho|A(f)|\tilde{h}^*_{\alpha_1}\dots\tilde{h}^*_{\alpha_n}\rangle\langle\tilde{h}_{\alpha_1}\dots\tilde{h}_{\alpha_n}|B(g)|\eta\rangle \tag{8}$$

for any $f(x), g(y) \in S(R^4)$, where $A(x), B(y)$ are some operator-valued generalized functions of the space $S(R^4)$, $|\eta\rangle, |\rho\rangle \in D'_S$, $\{h_\alpha(p)\}$ is a complete set of normalized functions from the space

$\widetilde{S}(\overline{\Omega}^+_\mu)$, with

$$\sum_a \tilde{h}_a(p)\tilde{h}^*_a(q) = 2p^0\delta(p-q), \tag{9}$$

$$(\tilde{h}_a, \tilde{h}_b) = \int d^4p\, \theta(p^0)\, \delta(p^2 - \mu^2) \tilde{h}_a(p) \tilde{h}_b^*(p) < \infty. \tag{10}$$

**Axiom IV. Existence and unitarity of the $S$-matrix.** There exists a unitary operator $S$, i.e. $SS^+ = S^+S = I$, considered as an isomorphism between two equipped Hilbert spaces of initial and final asymptotic states.

By virtue of the invariance of the description of the scattering process with respect to the proper Poincaré group $P_+^{\uparrow}$, it follows that the $S$-matrix must satisfy the condition $[U(a, \Lambda), S] = 0$ for any transformation $(a, \Lambda) \in P_+^{\uparrow}$.

**Axiom V. Stability.** The vacuum state $|0\rangle \in D_S$ and the unique generalized state $|\rho^{(1)}(p)\rangle \in S'^{(1)}(\overline{\Omega}_\mu^+)$ corresponding to the state of a real particle are invariant with respect to the $S$-matrix, i.e.

$$S|0\rangle = |0\rangle, \qquad S|\rho^{(1)}(p)\rangle = |\rho^{(1)}(p)\rangle. \tag{11}$$

**B. Axioms characterizing local properties.**
For practical purposes it will be convenient for us to use the representation of the $S$-matrix in the form:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \cdots \int \left( \prod_{j=1}^{n} d^4x_j \right) h_n(x_1, \dots, x_n) : \varphi_{\text{out}}(x_1) \dots \varphi_{\text{out}}(x_n) :; \tag{12}$$

$$K_x \varphi_{\text{out}}(x) \neq 0, \qquad K_x = \Box_x - \mu^2. \tag{13}$$

**Axiom VI. Functional structures of radiation operators.** The radiation operators

$$H^{(n)}(x_1, \dots, x_n) = \frac{\delta^n S}{\delta\varphi_{\text{out}}(x_1) \dots \delta\varphi_{\text{out}}(x_n)} S^+, \qquad n = 1, 2, \dots \tag{14}$$

a) exist in the sense of operator-valued generalized functions of the space $S'(R^{4n})$, $n = 1, 2, \dots$, i.e. to any functions $f^{(n)}(x_1, \dots, x_n) \in S(R^{4n})$, $n = 1, 2, \dots$, one can assign linear operators

$$H^{(n)}(f^{(n)}) = \int \cdots \int \left( \prod_{j=1}^{n} d^4x_j \right) f^{(n)}(x_1, \dots, x_n) \times$$

$$\times H^{(n)}(x_1, \dots, x_n), \quad n = 1, 2, \dots, \quad \text{in } D_S \tag{15}$$

with matrix elements

$$(\Phi, H^{(n)}(x_1, \dots, x_n)\Psi) \in S'(R^{4n}) \quad \text{for any } \Phi, \Psi \in D_S. \tag{16}$$

b) As operator-valued generalized functions of the space $S'(R^{4n})$ they are extendable to the space $S'^{\,\mathrm{KG}}_{+}(R^{4n})$ of sufficiently smooth solutions of the Klein-Gordon (KG) equation with positive energies, i.e. to any functions $f^{(n)}(x_1, \dots, x_n) = f_1(x_1) \dots f_n(x_n)$, where

$$f_j(x_j) = \frac{1}{(2\pi)^{3/2}} \int d^4p_j \, \theta(p_j^0)\delta(p_j^2 - \mu^2)\tilde{f}_j(p_j)e^{ip_{jx}j} \in S^{\mathrm{KG}}_{+}(R^4), \tag{17}$$

$\tilde{f}_j(p_j) \in S(\overline{\Omega}^{+}_{\mu})$, $j = 1, 2, \dots, n$, one can assign linear operators $\widehat{H}^{(n)}(f^{(n)})$ in $D_S$ with matrix elements $(\Phi, H^{(n)}(x_1, \dots, x_n)\Psi) \in S'^{\,\mathrm{KG}}_{+}(R^{4n})$, $n = 1, 2, \dots$, for any $\Phi, \Psi \in D_S$, and

$$(\Phi, \widehat{H}^{(n)}(f^{(n)})\Psi) = (\Phi(\widehat{H}^{(n)})(f^{(n)})\Psi), \qquad n = 1, 2, \dots, \tag{18}$$

for any $f^{(n)}(x_1, \dots, x_n) \in S^{\mathrm{KG}}_{+}(R^{4n})$ and for any $\Phi, \Psi \in D_S$.

**Axiom VII. Causality condition.** The causality condition in Bogoliubov’ s form is satisfied:

$$\iint d^4x \, dy \, f(x)g(y) \, \frac{\delta}{\delta\varphi(y)} \left( \frac{\delta S}{\delta\varphi(x)} S^{+} \right) = 0 \tag{19}$$

for any $f(x), g(y) \in S(R^4)$ with causally independent supports, i.e.

$$f(x)g(y) = 0 \quad \text{for } (y^0 - x^0) \geq 0 \text{ and } (y - x)^2 \geq 0. \tag{20}$$

**Axiom VIII. Reduction formulas.** The following reduction formulas hold:

$$[S, a^{+}(\tilde{g})] = \int d^4x \, \overset{*}{g}(x) \, \frac{\delta S}{\delta\varphi_{\mathrm{out}}(x)}; \tag{21}$$

$$[a(\tilde{g}), S] = \int d^4x \, g(x) \, \frac{\delta S}{\delta\varphi_{\mathrm{out}}(x)} \tag{22}$$

for any $\tilde{g}(p) \in S(\overline{\Omega}^{+}_{\mu})$, where the function $g(x) \in S^{\mathrm{KG}}_{+}(R^4)$ is defined by

$$g(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p \, \theta(p^0)\delta(p^2 - \mu^2)\tilde{g}(p)e^{ipx}. \tag{23}$$

Here $a^{+}(p)$ and $a(p)$ denote the creation and annihilation operators of particles of mass $\mu > 0$.

Using the reduction formulas, one can transform the matrix elements of the $S$-matrix

$$S_{mn}(p_1, \dots, p_m, -q_1, \dots, -q_n) =$$

$$= \left(\prod_{r=1}^{n} \theta(q_r^0)\delta(q_r^2 - \mu^2)\right)\left(\prod_{s=1}^{m} \theta(p_s^0)\delta(p_s^2 - \mu^2)\right) \times$$
$$\times \langle 0|a(p_1)\dots a(p_m) S a^+(q_1)\dots a^+(q_n)|0\rangle, \quad (24)$$

which are generalized functions from the space $S'(\Omega^+_{m+n})$, $\sum_{i=1}^{n} q_i = \sum_{j=1}^{m} p_j$, symmetric with respect to the four-momenta in $\{p_i, \dots, p_m\}$, $\{q_1, \dots, q_n\}$, and $L_+^\uparrow$-, $P$-, $C$-, and $T$-invariant, into vacuum averages of radiation operators.

Taking this opportunity, I express my deep gratitude to N. N. Bogoliubov, A. N. Tavkhelidze, I. T. Todorov, I. Byalynicki-Birula, B. V. Medvedev, and A. V. Efimov for useful discussions and valuable comments.

Joint Institute
for Nuclear Research

Received
27 II 1967

## CITED LITERATURE

1. N. N. Bogoliubov, B. V. Medvedev, M. K. Polivanov, *Problems in the Theory of Dispersion Relations*, Moscow, 1958.

2. N. N. Bogoliubov, D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Moscow, 1957.

3. S. Schweber, *Introduction to Relativistic Quantum Field Theory*, Moscow, 1963.

4. I. M. Gel' fand, N. Ya. Vilenkin, *Generalized Functions*, vol. 4, Moscow, 1961.

5. I. T. Todorov, International Winter School of Theoretical Physics at the Joint Institute for Nuclear Research, Dubna, 1963.

*Note: Figure translations are in progress. See original paper for figures.*