# DYNAMIC SYSTEMS WITH DELAY

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## Abstract

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# DYNAMIC SYSTEMS WITH DELAY

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In the present note autonomous systems of differential equations with delayed argument are studied from the point of view of the general theory of dynamical systems ([1]). This problem was posed in ([2]-[4]). Below, on the basis of the proposed axiomatization, the principal results in this direction are established, concerning invariant sets, minimal sets, and recurrent motions.

**1°.** In what follows, by the letter $\rho$ we shall denote distance in metric spaces. Let $R$ be a metric space; $T$ the half-line $[0, +\infty)$; $S$ a compact set contained in $T$; $\Phi$ a metric space whose elements are continuous mappings of the compact set $S$ into $R$, with the metric of uniform convergence on $S$; $f$ a continuous mapping $\Phi \times T$ into $R$. The ordered triple $(\Phi, T, f)$ will be called a **dynamical system with delay** if the following axioms are satisfied:

1) $f(\varphi, s) = \varphi(s)$ for all $\varphi \in \Phi$ and $s \in S$;
2) whatever $\varphi \in \Phi$ and $\tau \in T$ may be, the function $\varphi^\tau$, defined by the relation $\varphi^\tau(s) = f(\varphi, \tau + s)$ for all $s \in S$, belongs to $\Phi$;
3) whatever $t \in T$, $\tau \in T$, and $\varphi \in \Phi$ may be, the equality

$$f(\varphi, t + \tau) = f(\varphi^\tau, t).$$

The proposed axiomatization is quite general. Properties 1)–3) are possessed by solutions of the differential equation with delayed argument

$$x'(t) = g(x(t), x(t - \tau)), \qquad \text{where } 0 < \tau = \text{const},$$

provided that the right-hand side of the equation ensures existence, uniqueness, continuous dependence on the initial function, and continuability of the solution on $[\tau, +\infty)$. Properties 1)–3) are also possessed by solutions of the ordinary differential equation

$$x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_{n-1}(t)x' + p_n(t)x = 0,$$

satisfying the boundary conditions $x(a_k) = A_k,\ a_k \geq 0,\ k = 1, 2, \dots, n$, provided the conditions of existence, uniqueness, continuous dependence on the boundary conditions, and continuability of solutions for all $t \geq 0$ are fulfilled.

Finally, let us give an example of a dynamical system with delay defined in a space of continuous functions. Let $\Psi$ be some set of continuous functions defined on the half-line $T$ with values in the metric space $R$; $S$ a compact set contained in $T$; $\Phi$ the restrictions to $S$ of functions of the set $\Psi$. We shall regard the space $\Phi$ as metric with the metric of uniform convergence on $S$. Suppose that $\Psi$ satisfies the following conditions: a) if $\psi(t) \in \Psi$, then also $\psi(t+\tau) \in \Psi$ for any $\tau > 0$; b) if a sequence $\{\psi_n\}$ from $\Psi$ converges uniformly on $S$, then it converges uniformly on every segment contained in $T$. Define the mapping $f : \Phi \times T \to R$ by putting $f(\varphi, t) = \tilde{\varphi}(t)$, where $\tilde{\varphi}$ is the extension to $T$ of the function $\varphi$. We note that, by condition b), for each $\varphi \in \Phi$ the extension $\tilde{\varphi}$ is uniquely determined. Using the conditions

a) and b), it can be shown that the triple $(\Phi, T, f)$ constructed above is a dynamical system with delay.

2°. Let $(\Phi, T, f)$ be an arbitrary dynamical system with delay. Define the mapping $\sigma : \Phi \times T \to \Phi$ by setting $\sigma(\varphi, t) = \varphi^t$. It is easy to verify that the mapping $\sigma$ is continuous, $\sigma(\varphi, 0) = \varphi$, and $\sigma[\sigma(\varphi, t_1), t_2] = \sigma(\varphi, t_1 + t_2)$ for any $\varphi \in \Phi,\ t_1 \geq 0,\ t_2 \geq 0$. Thus it becomes possible to pass from the given dynamical system with delay to the Birkhoff ([1]) dynamical system $\sigma(\varphi, t)$, whose phase space is the set of initial functions $\Phi$. Such a passage is useful in a number of cases. The idea of passing from solutions of differential equations with delayed argument to trajectories in the space of initial functions is contained in ([5]).

**Theorem 1.** *For any $\varepsilon > 0,\ l > 0$, and $\varphi \in \Phi$, there exists a $\delta > 0$ such that if $\psi \in \Phi$ and $\rho(\varphi, \psi) < \delta$, then $\rho(f(\varphi, t), f(\psi, t)) < \varepsilon$ for all $0 \leq t \leq l$.*

Theorem 1 is a generalization of the property of integral continuity known in the general theory of dynamical systems.

A set $F \subseteq \Phi$ is called invariant in $\Phi$ if, for any $\varphi \in F$ and $t \geq 0$, the relation $\varphi^t \in F$ holds. Using the system $\sigma(\varphi, t)$ defined above, one can show that the union (intersection) of any collection of sets invariant in $\Phi$, as well as the closure of any set invariant in $\Phi$, is an invariant set in $\Phi$.

The function $f(\varphi, t)$, for fixed $\varphi \in \Phi$, will be called a motion, and the set of points $\{f(\varphi, t) : t \geq 0\}$ the trajectory of this motion. The trajectory of the motion $f(\varphi, t)$ will be denoted by the symbol $f(\varphi, T)$. Note that every finite arc of the trajectory $f(\varphi, T)$, as the continuous image of some segment from $T$, is compact.

A set $M \subseteq R$ is called invariant in $R$ if there exists a set $\Psi \subseteq \Phi$ such that

$$M = \bigcup_{\psi \in \Psi} f(\psi, T),$$

i.e., if $M$ consists of whole trajectories.

**Theorem 2.** *In order that a set $M \subseteq R$ be invariant in $R$, it is necessary and sufficient that there exist a set $F$, invariant in $\Phi$, such that $M = f(F, 0)$.*

We outline the proof. If the set $M$ is invariant in $R$, then

$$M = \bigcup f(\psi, T),$$

where $\Psi$ is some set from $\Phi$. The set

$$\{\psi^t : \psi \in \Psi,\ t \geq 0\}$$

is invariant in $\Phi$. In this case $M = f(F, 0)$. Conversely, if $M = f(F, 0)$, where $F$ is a set invariant in $\Phi$, then

$$M = \bigcup_{\varphi \in F} f(\varphi, T),$$

i.e., $M$ is invariant in $R$.

From the equality

$$\bigcup f(F_\alpha, t) = f\left(\bigcup F_\alpha, t\right),$$

valid for any $t \in T$ and $F_\alpha \subseteq \Phi$, and from Theorem 2, it follows that the union of any collection of sets invariant in $R$ is an invariant set in $R$. However, there are examples showing that the intersection of invariant sets in $R$ need not be an invariant set in $R$. One may also give an example of a set invariant in $R$ whose closure no longer has this property. Nevertheless, the following assertion is valid: if $M \subseteq R$ and $F \subseteq \Phi$ is a set such that

$$\bigcup_{\varphi \in F} f(\varphi, T) \subseteq M,$$

then

$$\bigcup_{\varphi \in \overline{F}} f(\varphi, T) \subseteq \overline{M}.$$

In particular, if $M$ is invariant in $R$, i.e. if

$$M = \bigcup_{\varphi \in F} f(\varphi, T),$$

where $F$ is some set from $\Phi$, then

$$\bigcup_{\varphi \in \overline{F}} f(\varphi, T) \subseteq \overline{M}.$$

What was noted above shows that, in their properties, dynamical systems with delay differ substantially from Birkhoff dynamical systems.

A dynamical system with delay $(\Phi, T, f)$ will be called **regular** if the following conditions are satisfied: 1) the compact $S$ consists-

contains zero; 2) from every sequence of initial functions $\{\varphi_n\}$ from $\Phi$, for which the corresponding sequence of points $\{\varphi_n(0)\}$ of the space $R$ converges, one can extract a subsequence converging in $\Phi$. In this definition condition 2) is essential. It is satisfied, for example, if the space $\Phi$ is compact or when $\Phi$ consists of constant functions. In particular, the regularity conditions are satisfied if $(\Phi, T, f)$ is a Birkhoff dynamical system (i.e., $\Phi$ consists of constant functions, and the compact set $S$ consists of a single point, coinciding with zero).

**Theorem 3.** *Suppose that the dynamical system $(\Phi, T, f)$ is regular. Then the closure of every invariant set in $R$ is invariant.*

We give the scheme of the proof. If $M$ is invariant in $R$, then on the basis of Theorem 2 the equality $M = f(F, 0)$ holds, where $F$ is some invariant set in $\Phi$. In this case $\overline{M} = f(\overline{F}, 0)$, whence the invariance of the closure $\overline{M}$ follows.

3°. A set $F \subseteq \Phi$ is called **minimal** in $\Phi$ if it is nonempty, closed, invariant, and contains no proper subset possessing these three properties.

**Theorem 4.** *In order that a nonempty invariant set $F \subseteq \Phi$ be minimal in $\Phi$, it is necessary and sufficient that, for every function $\varphi \in F$, the closure in $\Phi$ of the set $\{\varphi^t : t \geq 0\}$ coincide with $F$.*

**Theorem 5.** *The following assertions hold: 1) every nonempty invariant closed compact set $F \subseteq \Phi$ contains some minimal set in $\Phi$; 2) if the space $\Phi$ is compact, then it contains some minimal set in $\Phi$.*

The proof of Theorems 4 and 5 is carried out with the aid of the Birkhoff dynamical system $\sigma(\varphi, t)$ constructed in 2°. Its properties and its connection with the dynamical system with delay are used.

The motion $f(\varphi, t)$ is called **recurrent** if for every $\varepsilon > 0$ there exists $l > 0$ such that, whatever $t \geq 0$ may be, on every interval of length $l$ there is a $\tau$ for which

$$\rho(f(\varphi, t+s), f(\varphi, t+s+\tau)) < \varepsilon$$

for all $s \in S$.

**Theorem 6.** *If $\varphi$ belongs to a compact and minimal set in $\Phi$, then the motion $f(\varphi, t)$ is recurrent.*

**Theorem 7.** *Suppose that the space $R$ is complete. Then, if the motion $f(\varphi, t)$ is recurrent, the closure in the space $\Phi$ of the set $\{\varphi^t : t \geq 0\}$ is a compact and minimal set in $\Phi$.*

The last two theorems are a generalization of the fundamental results of the general theory of dynamical systems known as Birkhoff' s theorems (see ([1]), pp. 402–404).

In the study of dynamical systems with delay, along with sets minimal in $\Phi$, it makes sense also to consider sets minimal in $R$. In the case of Birkhoff systems these two notions coincide.

A set $M \subseteq R$ is called **minimal** in $R$ if it is nonempty, closed, invariant, and contains no proper subset possessing the same properties.

**Theorem 8.** *Suppose that the dynamical system with delay* $(\Phi, T, f)$ *is regular. Then, in order that a nonempty invariant set* $M \subseteq R$ *be minimal in* $R$*, it is necessary and sufficient that, for any trajectory* $f(\varphi, T)$ *contained in* $M$*, the equality*

$$\overline{f(\varphi, T)} = M$$

*hold.*

The proof of this theorem is similar to the proof of the analogous assertion from ([1]).

**Theorem 9.** *Suppose that the dynamical system with delay* $(\Phi, T, f)$ *is regular, and that the set* $M \subseteq R$ *is compact and minimal*

in $R$. Then there is a function $\varphi \in \Phi$ such that $\tilde{f}(\varphi, T) = M$, and the motion $f(\varphi, t)$ is recurrent.

The proof is based on Theorems 5, 6, and 8. We note that in Theorems 8 and 9 the regularity condition for the system $(\Phi, T, \tilde{f})$ is essential.

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*Note: Figure translations are in progress. See original paper for figures.*