GENERAL INTEGRAL REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract

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I. I. BAVRIN

GENERAL INTEGRAL REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Bergerskii ([1,2]) used, in constructing integral representations, the constructions of kernels given in the integral representations of Cauchy ($n \geq 1$) and Temlyakov. This enabled him to obtain, for functions $f(z)$ ($z = (z_1, \dots, z_n)$), holomorphic in complete n-circular ($n \geq 2$) domains, integral representations over $(n+p)$-dimensional manifolds ($p = 0, 1, \dots, n-1$) on the $(2n-1)$-dimensional boundary of the domain, in which the kernels with respect to the coordinates of the point z are holomorphic functions. Below, as follows from the specific properties of complete n-circular domains, it is noted (see § 2) that more general integral representations are also valid, including, in particular, the integral formulas from ([1]). In the exposition of § 2 we adhere to the definitions and notation used in ([1]).

§ 1. Let $f = f(z_1, \dots, z_n)$ be a function holomorphic in a complete n-circular ($n \geq 2$) domain Q with center at the point $(0, \dots, 0)$, k a natural number, and $\gamma_1, \dots, \gamma_k$ arbitrary positive numbers satisfying $\gamma_j \geq 1$ ($j = 1, \dots, k$). Further, for each j from the set $\{1, \dots, k\}$, let $\delta_1^{(j)}, \dots, \delta_n^{(j)}$ be arbitrary nonnegative numbers satisfying $\delta_1^{(j)} + \dots + \delta_n^{(j)} > 0$. Introduce the notation

$$L_{A_j}[f] \equiv L_{(\gamma_j, \delta_1^{(j)}, \dots, \delta_n^{(j)})}[f] = \gamma_j f + \sum_{\nu=1}^{n} \delta_\nu^{(j)} z_\nu f'_\nu, \quad (f'_\nu \equiv f'_{z_\nu}), \quad j = 1, \dots, k,$$

$$L_A^{(k)}[f] \equiv L_{(A_1, \dots, A_k)}^{(k)}[f] = L_{A_k}[L_{A_{k-1}} \cdots [L_{A_1}[f]] \cdots]^{*},$$

$$L_A^{(-k)}[f] = \int_0^1 d\varepsilon_1 \cdots \int_0^1 d\varepsilon_{k-1} \int_0^1 \varepsilon_1^{\gamma_1 - 1} \cdots \varepsilon_k^{\gamma_k - 1} f(\varepsilon_1^{\delta_1^{(1)}} \cdots \varepsilon_k^{\delta_1^{(k)}} z_1, \dots$$

$$\dots, \varepsilon_1^{\delta_n^{(1)}} \cdots \varepsilon_k^{\delta_n^{(k)}} z_n)\, d\varepsilon_k.^{**}$$

and put $L_A^{(0)}[f] = f$.

Theorem 1. *If the function $f(z)$ $(n \geq 2)$ is holomorphic in the domain Q, then for every natural k the formula*

$$f(z) = \int_0^1 d\varepsilon_1 \cdots \int_0^1 d\varepsilon_{k-1} \int_0^1 \varepsilon_1^{\gamma_1 - 1} \cdots \varepsilon_k^{\gamma_k - 1} L_A^{(k)} [f(\varepsilon_1^{\delta_1^{(1)}} \cdots \varepsilon_k^{\delta_1^{(k)}} z_1, \dots$$

$$\dots, \varepsilon_1^{\delta_n^{(1)}} \cdots \varepsilon_k^{\delta_n^{(k)}} z_n)]\, d\varepsilon_k. \tag{1}$$

holds in the domain Q.

The proof is similar to the proof of Theorem 1 in the author' s note ([3]). With the aid of formula (1) it is established that

$$L_A^{(-k)}[L_A^{(k)}[f]] = L_A^{(k)}[L_A^{(-k)}[f]] = f.$$

* Here, as everywhere below, $A_j = (\gamma_j, \delta_1^{(j)}, \dots, \delta_n^{(j)})$, $j = 1, \dots, k$, $A = (A_1, \dots, A_k)$ $(k \geq 1)$.

** Here and everywhere below we assume $0^0 = 1$.

Consequently, $L_A^{(-k)}[f]$ is the inverse operator with respect to the operator $L_A^{(k)}[f]$ (or, briefly, the inverse operator).

Only the operators just introduced are used below in the propositions of § 2 (for explanations concerning some of the operators introduced earlier by the author, also used in the propositions of § 2, see (3-6)).

Remark 1. Under the hypothesis of Theorem 2 from the author' s note (7), formula (2) in that theorem also holds in the case when, in formula (2), Y is replaced by A. The proof is unchanged.

§ 2. In what follows, a is a number equal to 0 or 1; D is a bounded complete n-circular $(n \geq 2)$ domain with center at the origin; Δ is the same bounded convex complete n-circular $(n \geq 2)$ domain with center at the origin that occurs in (1) (see (1), pp. 166, 167); $\overline{D}$ and $\overline{\Delta}$ are closed domains.

Theorem 2. Let the function $f(z)$ $(n \geq 2)$ be holomorphic in D. Then, if the functions $f_\nu^{(\alpha)}(z)$, $\nu = 1, \dots, n$, and all their partial derivatives up to order μ $(\mu \geq 0)$ inclusive are continuous in $\overline{D}$, then for every point $z \in D$ there exists a boundary point $\eta \in \partial D$ such that $|z_\nu| < |\eta_\nu|$ $(\nu = 1, \dots, n)$ and, for $p = 1, \dots, n-1$, $k = 0, 1, \dots, \mu$,

$$f(z) = \alpha f(0) + \frac{1}{n + \alpha(1-n)} \sum_{\nu=1}^{n} \frac{z_\nu^\alpha}{(2\pi i)^n} \int_{C_1} \frac{d\xi_1}{\xi_1} \cdots \tag{2}$$

$$\cdots \int_{C_n} \frac{d\xi_n}{\xi_n} \int_{S_p} L_{1,\alpha}^{(-\alpha)} \left[L_{n-p,n-1}^{(p)} \left[L_A^{(-k)}[I] \right] \right] L_A^{(k)} \left[f_\nu^{(\alpha)}(\zeta) \right] (1-\tau_p)^{n-p-1}\, d\tau_1 \cdots d\tau_p^{*},$$

where $\zeta = (\xi_1, \dots, \xi_n)$, C_ν $(\nu = 1, \dots, n)$ is the circle $|\xi_\nu| = |\eta_\nu|$; S_p is the p-dimensional $(p = 1, \dots, n-1)$ simplex (see (1), p. 165),

$$I = \prod_{l=p+1}^{n} (1 - U_{pl})^{-1},$$

$$U_{pl} = \frac{\tau_1}{\xi_1} z_1 + \frac{\tau_2 - \tau_1}{\xi_2} z_2 + \cdots + \frac{\tau_p - \tau_{p-1}}{\xi_p} z_p + \frac{1 - \tau_p}{\xi_l} z_l.$$

Theorem 3. Let the function $f(z)$ $(n \geq 2)$ be holomorphic in Δ. Then, if the functions $f_\nu^{(\alpha)}(z)$, $\nu = 1, \dots, n$, and all their partial derivatives up to order μ $(\mu \geq 0)$ inclusive are continuous in $\overline{\Delta}$, then, for a point $z \in \Delta$, $p = 1, \dots, n-1$, $k = 0, 1, \dots, \mu$,

$$f(z) = \alpha f(0) + \frac{1}{n + \alpha(1-n)} \sum_{\nu=1}^{n} \frac{z_\nu^\alpha}{(2\pi)^n} \int_0^{2\pi} d\varphi_1 \cdots$$

$$\cdots \int_0^{2\pi} d\varphi_n \int_{T_p} L_{1,\alpha}^{(-\alpha)} \left[L_{n-p,n-1}^{(p)} \left[L_A^{(-k)}[J_p] \right] \right] L_A^{(k)} \left[f_\nu^{(\alpha)}(r_1(\tau) e^{i\varphi_1}, \dots \right.$$

$$\left. \dots, r_n(\tau) e^{i\varphi_n}) \right] (1-\tau_p)^{n-p-1}\, d\tau_1 \cdots d\tau_p^{**}, \qquad (3)$$

* In the case $p = 0$ formula (2) must be replaced by the following:

$$f(z) = \alpha f(0) + \frac{1}{n + \alpha(1-n)} \sum_{\nu=1}^{n} \frac{z_\nu^\alpha}{(2\pi i)^n} \int_{C_1} \frac{d\xi_1}{\xi_1} \cdots$$

$$\cdots \int_{C_n} L_{1,\alpha}^{(-\alpha)} \left[L_A^{(-k)} \left[\prod_{l=1}^{n} \left(1 - \frac{z_l}{\xi_l} \right)^{-1} \right] \right] L_A^{(k)} \left[f_\nu^{(\alpha)}(\zeta) \right] \frac{d\xi_n}{\xi_n} \qquad (a)$$

** In the case $p = 0$ formula (3) must be replaced by a formula analogous in form to formula (a).

where T_p is the p-dimensional set* defined in (1) (p. 167),

$$J_p = \prod_{l=p+1}^{n} (1-u_{pl})^{-1}, \qquad u_{pl} = \frac{\tau_1}{r_1(\tau)} e^{-i\varphi_1} z_1 +$$

$$+\frac{\tau_2-\tau_1}{r_2(\tau)} e^{-i\varphi_2} z_2 + \dots + \frac{\tau_p-\tau_{p-1}}{r_p(\tau)} e^{-i\varphi_p} z_p + \frac{1-\tau_p}{r_l(\tau)} e^{i\varphi_l} z_l$$

(the further explanations possible here are omitted).

Corollary 1. Let the function $f(z)$ $(n \geq 2)$ be holomorphic in D, and

$$\alpha_p = \begin{cases} 0 \text{ for } p = 1, \dots, n-2 \ (n > 2), \\ \alpha \text{ for } p = n-1 \ (n \geq 2). \end{cases}$$

Then, if the functions $f_\nu^{(\alpha_p)}(z)$, $\nu = 1, \dots, n$, and all their partial derivatives up to order μ $(0 \leq \mu \leq p-\alpha_p)$, inclusive, are continuous in $\overline{D}$, then for every point $z \in D$ there exists a point $\eta \in \partial D$ such that $|z_\nu| < |\eta_\nu|$ $(\nu = 1, \dots, n)$, and, for $p = 1, \dots, n-1;\ k = 0, 1, \dots, \mu$,

$$f(z) = \alpha_p f(0) + \frac{1}{n+\alpha_p(1-n)} \sum_{\nu=1}^{n} \frac{z_\nu^{\alpha_p}}{(2\pi i)^n} \int_{C_1} \frac{d\zeta_1}{\zeta_1} \cdots$$

$$\cdots \int_{C_n} \frac{d\zeta_n}{\zeta_n} \int_{S_p} L^{(p-k-\alpha_p)}_{\binom{m_{\alpha_p+n-p}}{m_{n-k-1}}}[I]\, L^{(k)}_{\binom{m_{n-k}}{m_{n-1}}}[f_\nu^{(\alpha_p)}(\zeta)](1-\tau_p)^{n-p-1}\, d\tau_1 \cdots d\tau_p,$$

where $m_{\alpha_p+n-p}, m_{\alpha_p+n-p+1}, \dots, m_{n-1}$ $(n \geq 3)$ are arbitrary, but all distinct natural numbers taken from $\{\alpha_p+n-p, \alpha_p+n-p+1, \dots, n-1\}$, and for $n = 2, \alpha = 0$, $m_1 = 1$.

Remark 2. Along with the point $z \in D$ indicated in Theorem 2, the integral formula (2) will also hold in the polydisc $\{|z_\nu| < |\eta_\nu|,\ \nu = 1, \dots, n\}$. Analogously also in the case of Corollary 1.**

Corollary 2. Let the function $f(z)$ $(n \geq 2)$ be holomorphic in Δ, and

$$\alpha_p = \begin{cases} 0 \text{ for } p = 1, \dots, n-2 \ (n > 2), \\ \alpha \text{ for } p = n-1 \ (n \geq 2). \end{cases}$$

Then, if the functions $f_\nu^{(\alpha_p)}(z)$, $\nu = 1, \dots, n$, and all their partial derivatives up to order μ $(0 \leq \mu \leq p-\alpha_p)$, inclusive, are continuous in Δ, then for a point $z \in \Delta$, $p = 1, \dots, n-1;\ k = 0, 1, \dots, \mu$,

$$f(z) = \alpha_p f(0) + \frac{1}{n+\alpha_p(1-n)} \sum_{\nu=1}^{n} \frac{z_\nu^{\alpha_p}}{(2\pi)^n} \int_0^{2\pi} d\varphi_1 \cdots$$

$$\cdots \int_0^{2\pi} d\varphi_n \int_{T_p} L^{(p-k-\alpha_p)}_{\binom{m_{\alpha_p+n-p}}{m_{n-k-1}}}[J_p]\, L^{(k)}_{\binom{m_{n-k}}{m_{n-1}}}[f_\nu^{(\alpha_p)}(r_1(\tau)e^{i\varphi_1}, \dots$$

$$\dots, r_n(\tau)e^{i\varphi_n})](1-\tau_p)^{n-p-1}\, d\tau_1 \cdots d\tau_p.$$

Remark 3. Setting in Corollaries 1 and 2 $\alpha = 0$, $m_{n-p} = n-p$, $m_{n-p+1} = n-p+1, \dots, m_{n-1} = n-1$, we obtain formulas that were earlier obtained, in another form, by Bierski ([1]).

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* For $p < n-1$, $T_p \subset S_{n-1}$, and for $p = n-1$, $T_p = S_{n-1}$ ([1]).

** If the domain D is a polydisc $\{|z_\nu| < R_\nu,\ \nu = 1, \dots, n\}$, then for all points $z \in D$ there exists one and the same point η (as it one may take any point of the skeleton of the given polydisc).

Remark 4. The integral formulas in Corollaries 1, 2, similarly to those noted in (8) (item 2), are solutions of the problems of establishing, in the case $n = 3$, the corresponding general integral representations and of their complete extension to the case $n > 3$.

Remark 5. Formula (1) (for $k = 1$) remains valid also in the case when γ_1 is any positive number. But for $0 < \gamma_1 < 1$, the integral entering this formula should be understood as improper. Taking into account the same remark concerning analogous integrals, Theorem 1 and all the content of the present article connected with $\gamma_1, \dots, \gamma_k$ remain valid also in the case when $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers.

Moscow Regional Pedagogical Institute
named after N. K. Krupskaya

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