UNIQUENESS CLASSES FOR THE SOLUTION OF A BOUNDARY-VALUE PROBLEM IN AN INFINITE STRIP

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 517.944

MATHEMATICS

V. M. BOROK

UNIQUENESS CLASSES FOR THE SOLUTION OF A BOUNDARY-VALUE PROBLEM IN AN INFINITE STRIP

(Presented by Academician I. G. Petrovskii, 22 IV 1968)

It is well known (see, for example, ([1])) that the study of the process of heat conduction in a finite rod at a sufficiently remote time from the initial instant leads to solving a problem without initial conditions for the heat equation:

$$\partial u(x,t)/\partial t = \partial^2 u(x,t)/\partial x^2, \qquad 0 < x < X, \qquad -\infty < t < \infty$$

with boundary conditions, for example, of the form $u(0,t) = u_0(t)$, $u(x,t) = u_1(t)$, $-\infty < t < \infty$; such a problem has a unique solution in the class of bounded functions. In the present note, uniqueness and nonuniqueness classes for the solution of similar problems in an infinite strip are established for arbitrary equations of the form

$$L(\partial/\partial x, \partial/\partial t)u(x,t) \equiv \partial^2 u/\partial t^2 + P(\partial/\partial x)\partial u/\partial t + Q(\partial/\partial x)u = 0, \qquad (1)$$

$$0 < t < T, \qquad x = (x_1, \dots, x_n), \qquad -\infty < x_i < \infty, \qquad i = 1, \dots, n,$$

where $P(\partial/\partial x)$, $Q(\partial/\partial x)$ are polynomials in $\partial/\partial x_1, \dots, \partial/\partial x_n$ with constant (complex-valued) coefficients. It is clarified under what conditions (described in terms of growth as $|x| \to \infty$) the only solution of equation (1) satisfying the zero boundary conditions

$$u(x,0) = u(x,T) = 0, \qquad -\infty < x_i < \infty, \qquad i = 1, \dots, n, \qquad (2)$$

can be the function $u(x,t) \equiv 0$.

1°. The main results of the paper are a concretization of the following general theorem. Let Φ be a linear topological space of functions $\varphi(x)$, $x = (x_1, \dots, x_n)$, defined for $-\infty < x_i < \infty$, $i = 1, \dots, n$, E a normed space, E' the dual space, with $\Phi \subset E$ and Φ dense in E.

Theorem 1. Suppose: 1) for every function $\varphi(x) \in \Phi$ the "adjoint problem"

$$L^*(\partial/\partial x, \partial/\partial t)v(x,t) \equiv L(-\partial/\partial x, -\partial/\partial t)v(x,t) = 0, \tag{1'}$$

$$v(x,0) = 0, \qquad v(x,T) = \varphi(x) \tag{2'}$$

has a solution $v(x,t)$, and for every t, $0 < t < T$, $v(x,t) \in E$;
2) the Cauchy problem $Lu = 0$, $u(x,T) = u_t'(x,T) = 0$, $u(x,t) \in E'$, has only the trivial solution.

Then every solution of problem (1)–(2) which belongs to the space E' for each $t \in [0,T]$ is identically equal to zero.

2°. Let $\Phi = \{\varphi(x)\}$ be a linear topological space of basic functions, and let $\Psi = \{\psi(s)\}$ be the space of Fourier transforms of functions from Φ ([2]). Then problem $(1')$–$(2')$ in the space Ψ passes into the following problem: solve the equation

$$L(-is, -\partial/\partial t)y(s,t) = 0, \qquad s = \sigma + i\tau, \tag{1''}$$

under the conditions

$$y(s,0) = 0, \qquad y(s,T) = \psi(s). \tag{2''}$$

Here $y(s,t)$ is the Fourier transform (in x) of the function $v(x,t)$, and $\psi(s)$ is the Fourier transform of the function $\varphi(x)$. The solution of problem $(1'')$–$(2'')$ has the form

$$y(s,t) = R(s,t)\psi(s), \tag{3}$$

where $R(s,t)$ is a function meromorphic (for each $t \in (0,T)$) with respect to s, having the form

$$R(s,t) = \begin{cases} \dfrac{\exp\{\lambda_1(s)t\} - \exp\{\lambda_2(s)t\}}{\exp\{\lambda_1(s)T\} - \exp\{\lambda_2(s)T\}}, & \text{if } \lambda_1(s) \neq \lambda_2(s), \\ t/T \exp\{\lambda_1(s)(t-T)\}, & \text{if } \lambda_1(s) = \lambda_2(s). \end{cases} \tag{4}$$

Here $\lambda_1(s), \lambda_2(s)$ are the roots of the equation

$$\lambda^2 - P(-is)\lambda + Q(-is) = 0.$$

It is seen from (4) that the poles of the function $R(s,t)$ are those points s at which the polynomial

$$D(s) = \tfrac{1}{4}P^2(-is) - Q(-is)$$

takes the values $-k^2\pi^2T^{-2}$, $k = 1, 2, \ldots$. We denote this set by Z and put

$$a = \inf_{s\in Z} |\operatorname{Im} s|. \tag{5}$$

Let first $a > 0$. Then in the domain $|\operatorname{Im} s| < a$ the function $R(s,t)$ is analytic for each $t \in (0,T)$ and, for $|\operatorname{Im} s| \le a_1 < a,\ t \in [0,T]$, satisfies the estimate

$$|R(s,t)| \le C_1 \exp\{C_2|s|^p\},$$

where p is the degree of the polynomial $P(s)$. Then, by means of Cauchy’ s formula, one can obtain the estimate

$$\left|\partial^q R(\sigma,t)/\partial\sigma_1^{q_1}\cdots\partial\sigma_n^{q_n}\right| \le C_3(ea_1)^q \prod_{j=1}^{n} q_j^{q_j} \exp\{c|\sigma|^p\}, \tag{6}$$

$$q = q_1 + \cdots + q_n, \qquad -\infty < \sigma_i < \infty, \quad i = 1, 2, \ldots, n.$$

Estimate (6) shows (see (2), pp. 281–283) that multiplication by the function $R(\sigma,t)$ is a bounded operation, defined in the space

$$S^{1,B}_{1/p,A} = S^{1,\ldots,1;\,B,\ldots,B}_{1/p,\ldots,1/p,A,\ldots,A}$$

for sufficiently small A, and mapping it into the space

$$S^{1,B'}_{1/p,A'},$$

where

$$B' = B + (a_1e)^{-1}, \qquad A'^{-p} = A^{-p} - C_4pe.$$

Thus, for any $\psi(s) \in S^{1,B}_{1/p,A}$ the solution of problem (1″)–(2″) can be found by formula (3) and is a function in the space

$$S^{1,B'}_{1/p,A'}.$$

Applying the inverse Fourier transform, we obtain that problem (1′)–(2′), for any $\varphi(x) \in S^{1/p,A}_{1,B}$, has a solution $v(x,t)$, with

$$v(x,t) \in S^{1/p,A'}_{1,B'}.$$

Since, as B, one may take any positive number, $B' > (a_1e)^{-1}$ is arbitrary.

As the normed space E we choose the space of functions $f(x)$ for which

$$\|f(x)\| = \int \cdots \int |f(x)| \exp\{b|x|\}\, dx_1 \cdots dx_n < \infty.$$

Here b is a fixed number such that $0 < b < a$. We now find a_1 from the condition $b < a_1 < a$, and the number B' from the condition $(ea_1)^{-1} < B' < (be)^{-1}$. Then

$$S_{1,B'}^{1/p,A'} \subset E.$$

Thus, for any function $\varphi(x) \in S_{1,B}^{1/p,A}$, the solution $v(x,t)$ of problem $(1')$–$(2')$ exists and belongs to the space E. Moreover,

$$S_{1,B}^{1/p,A} \subset E,$$

and, since the space $S_{1,B}^{1/p,A}$ is nontrivial, it is dense in E ((2), pp. 278–280, 288). Finally, the Cauchy problem $Lu = 0$, $u(x,T) = u_t'(x,T) = 0$ can have in E' only the solution $u(x,t) \equiv 0$ (3). Applying Theorem 1, we arrive at the following result.

Theorem 2. Let the number a, defined in (5), be positive. Then in the class of functions satisfying the estimate

$$|u(x,t)| \leq A_0 \exp\{b|x|\},$$

$$x = (x_1, \dots, x_n), \qquad -\infty < x_i < \infty, \quad i = 1, \dots, n, \quad 0 < t < T, \tag{7}$$

for $b < a$, problem (1)–(2) can have only the trivial solution.

3°. **Theorem $2'$.** Let $a > 0$. Then in the class of functions satisfying the estimate (7) with $b > a$, problem (1)–(2) has a nontrivial solution.

If, moreover, it is known that there exists $s_a \in Z$ such that $|\operatorname{Im} s_a| = a$, then in the class of functions satisfying (7) with $b = a$, there is also a nontrivial solution of problem (1)–(2).

Indeed, let $s_b \in Z$ and $|\operatorname{Im} s_b| \leq b$. The function

$$u(x,t) = \exp\{-i(s_b, x)\} \left[\exp\{-\lambda_1(s_b)t\} - \exp\{-\lambda_2(s_b)t\}\right] \tag{8}$$

is a nontrivial solution of problem (1)–(2) and satisfies condition (7). The same example, with s_b replaced by s_a, establishes the validity of the second assertion of the theorem.

Let us note that $a = \infty$ if and only if $D(s) \equiv \text{const}$. In this case the function $R(s,t)$ is entire and has the form

$$R(s,t) = A(t)\exp\left\{\frac{P(-is)}{2}(t-T)\right\}.$$

Analogously to how this is done in the theory of uniqueness classes for the solution of the Cauchy problem ((3), pp. 63–66), one can prove that in this case problem (1)–(2) has only the trivial solution in the class M_α of functions $f(x)$ satisfying the estimate

$$|f(x)| \le C_\varepsilon \exp\{\varepsilon|x|^\alpha\}, \qquad \varepsilon > 0, \tag{7'}$$

where $\alpha = p'$, $p^{-1} + {p'}^{-1} = 1$, $p > 1$, is the degree of the polynomial $P(s)$, and has a nontrivial solution in the class M_α if $\alpha \supset p'$. If $p = 1$, then the solution of problem (1)–(2) is trivial in the class of functions M_α for every $\alpha > 0$.

4°. **Lemma.** If $|\operatorname{Im} s| > 0$ for all $s \in Z$, then for some $\mu > 0$ and $C > 0$ the domain

$$\Omega = \{s: \ s = \sigma + i\tau = (\sigma_1 + i\tau_1, \dots, \sigma_n + i\tau_n),\ |\tau| \le C(1+|\sigma|)^{-\mu}\} \tag{9}$$

does not contain points of the set Z.

Thus, if $a = 0$, but $|\operatorname{Im} s| > 0$ for all $s \in Z$, then the poles of the function $R(s,t)$ are located outside the domain (9). Estimating again the functions $\partial^q R(\sigma,t)/\partial\sigma_1^{q_1}\cdots\partial\sigma_n^{q_n}$ by means of the Cauchy formula and choosing in the latter, as the contour of integration, the boundary of a circular polycylinder with center at the point $(\sigma_1, \dots, \sigma_n)$, lying entirely in the domain Ω, one can obtain the estimate

$$\left|D^{(q_1,\dots,q_n)}R(\sigma,t)\right| \le A_2 \prod_{j=1}^{n}\left(q_j^{\xi q_j} A_{3j}^{q_j}\right)\exp\{A_4|\sigma|^l\}. \tag{10}$$

Here $\beta > 1$, $A_3 > 0$, $A_4 > 0$; by increasing A_4 and l, one can make $\beta - 1$ and A_{3j}, $j = 1, \dots, n$, arbitrarily small. Estimate (10) shows ((2), pp. 281–283) that multiplication by $R(\sigma,t)$ is a bounded operation in the space $S_{1/l,A}^{\beta,B}$, mapping it into the space $S_{1/l,A'}^{\beta,B+A_3}$ ($A' > A$ and is expressed in terms of A_4 and l). Then for any $\psi(s) \in S_{1/l,A}^{\beta,B}$ problem $(1'')$–$(2'')$ has a solution $y(s,t) \in S_{1/l,A'}^{\beta,B+A_3}$. Hence it follows that problem $(1')$–$(2')$, for any $\varphi(x) \in S_{\beta,B}^{1/l,A}$, has a solution $v(x,t) \in S_{\beta,B+A_3}^{1/l,A'}$.

In this case, as the normed space E, we choose the space of functions $f(x)$ for which

$$\|f(x)\| = \int \cdots \int |f(x)| \exp\{b_1 |x|^{b_2}\} \, dx_1 \cdots dx_n < \infty.$$

Here $b_1 > 0,\ 0 < b_2 < 1$ are arbitrary (fixed) numbers. Choose A_4 and l in (10), and also B, so that the conditions $\beta = b_2^{-1}$, $\beta e^{-1}(B + A_3)^{-1/\beta} > b_1$ are satisfied. Then the embeddings $S_{\beta,B}^{1/l,A} \subset S_{\beta,B+A_3}^{1/l,A'} \subset E$ hold, and $S_{\beta,B}^{1/l,A}$ is dense in E ((2), pp. 278-280, 288).

Applying Theorem 1, we obtain the following result.

Theorem 3. Suppose $|\operatorname{Im} s| > 0$ for all $s \in Z$ and

$$\inf_{s \in Z} |\operatorname{Im} s| = 0.$$

Then, for any $b > 0$ and $0 < c < 1$, in the class of functions satisfying the estimate

$$|u(x,t)| \le A_0 \exp\{b|x|^c\}, \qquad -\infty < x_i < \infty, \quad i = 1, \dots, n, \quad 0 < t < T,$$

problem (1)–(2) can have only the trivial solution.

5°. **Theorem 3′.** Suppose $|\operatorname{Im} s| > 0$ for all $s \in Z$ and

$$\inf_{s \in Z} |\operatorname{Im} s| = 0.$$

Then, for any $\varepsilon > 0$, in the class of functions

$$|u(x,t)| \le A_0 \exp\{\varepsilon|x|\}, \qquad -\infty < x_i < \infty, \quad i = 1, \dots, n, \quad 0 < t < T,$$

problem (1)–(2) has a nontrivial solution.

If $s_\varepsilon \in Z$ and $|\operatorname{Im} s_\varepsilon| \le \varepsilon$, then, replacing s_b in (8) by s_ε, we obtain a confirming example.

6°. **Theorem 4.** If there exists $s_0 \in Z,\ \operatorname{Im} s_0 = 0$, then problem (1)–(2) has a nontrivial bounded solution. If a solution of problem (1)–(2) belongs to the class $L_2(-\infty, \infty)$ or $L_1(-\infty, \infty)$, then it is identically equal to zero.

The validity of the first assertion of the theorem is proved by example (8), in which s_b should be replaced by s_0.

Assuming that the solution $u(x,t)$ of problem (1)–(2) belongs to $L_2(-\infty, \infty)$ or $L_1(-\infty, \infty)$, we obtain that its Fourier transform (with respect to x) $\tilde{u}(\sigma, t)$ must be equal to zero almost everywhere, whence the result follows.

As examples, let us consider the classical equations

$$\text{a)} \quad \partial^2 u/\partial t^2 = c\,\partial u/\partial x, \quad c = \text{const}; \qquad \text{b)} \quad \partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 = 0;$$

$$\text{c)} \quad \partial^2 u/\partial t^2 + \partial^2 u/\partial x^2 = 0.$$

For equation a) in the case $\operatorname{Re} c = 0$, and for equation b), problem (1)–(2) has nontrivial solutions in the class of bounded functions; for equation a) in the case $\operatorname{Re} c \neq 0$, the number a (see (5)) has the form

$$a = \pi^2 |\operatorname{Re} c| T^{-2} |c|^{-2},$$

and Theorems 2 and 2′ are applicable. In particular, for the heat equation $\partial^2 u/\partial t^2 = \partial u/\partial x$, the solution of problem (1)–(2) is trivial in the class of functions

$$|f(x)| \leq C \exp\{(\pi^2/T^2 - \varepsilon)|x|\}, \qquad \varepsilon > 0,$$

and nontrivial in the class of functions

$$|f(x)| \leq C \exp\{\pi^2 T^{-2} |x|\}.$$

For equation c), $a = \pi/T$; the solution is trivial in the class

$$|f(x)| \leq C \exp\{(\pi/T - \varepsilon)|x|\}, \qquad \varepsilon > 0,$$

and nontrivial in the class

$$|f(x)| \leq C \exp\{\pi T^{-1} |x|\}.$$

Kharkov State University
named after A. M. Gorky

Received
22 IV 1968

REFERENCES

1. A. N. Tikhonov, A. A. Samarskii, *Equations of Mathematical Physics*, Moscow-Leningrad, 1951.
2. I. M. Gel' fand, G. E. Shilov, *Generalized Functions*, vol. 2, *Spaces of Test and Generalized Functions*, Moscow, 1958.
3. I. M. Gel'fand, G. E. Shilov, *Generalized Functions*, vol. 3, *Some Questions in the Theory of Differential Equations*, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.