SIMPLE MALCEV ALGEBRAS OVER A FIELD OF CHARACTERISTIC ZERO

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Abstract

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SIMPLE MALCEV ALGEBRAS OVER A FIELD OF CHARACTERISTIC ZERO

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A description is obtained of the central simple Malcev algebras over an arbitrary field F of characteristic 0. Apart from the simple Lie algebras, this class is exhausted by a series of 7-dimensional algebras connected with the Cayley-Dickson algebras over F. For the algebras of this series the isomorphism problem is solved; in particular, all (finite-dimensional) simple Malcev algebras over the field of real numbers are described.

Malcev algebras (ML-algebras) were first defined in (^{1}) as algebras satisfying the identities

$$x^2 = 0,$$

$$J(x, y, xz) = J(x, y, z)x,$$

where $J(x, y, z) = (xy)z + (yz)x + (zx)y$. This class of algebras is a natural generalization of the class of Lie algebras. Malcev algebras are closely connected with a number of other algebraic objects: alternative algebras, analytic Moufang loops, and ternary Lie systems. At present the theory of these algebras is being successfully developed ($^{4-15}$), and many facts of the theory of Lie algebras are naturally generalized within the framework of the theory of Malcev algebras.

One of the basic questions in constructing the structure theory of any class of algebras is, as a rule, the problem of classifying the (finite-dimensional) simple algebras of the given class. Applied to the class of Malcev algebras, it is natural to pose the question of finding all simple Malcev algebras that are not simple Lie algebras. Until now this problem had been solved only in the case where the ground field F is algebraically closed and has characteristic 0 (^{6}). In this case it turned out that, up to isomorphism, there exists only one simple Malcev algebra A of this type; it has dimension 7 over F, and a basis e_i ($i = 1, \dots, 7$) in A can be chosen in such a way that the multiplication table of A has the form

$$e_i e_j = e_k, \qquad e_j e_k = e_i, \qquad e_k e_i = e_j, \tag{1}$$

where $(i,j,k) = (1,2,3), (1,4,5), (1,6,7), (2,4,6), (2,7,5), (3,7,4), (3,6,5)$. We note that this result was obtained under a certain additional restriction on A, which turned out to be inessential, as follows from the works ([9], [10], [13]).

Using the multiplication table (1), one can obtain a certain series of central simple Malcev algebras over an arbitrary field F (even in the case of finite characteristic). Let $\alpha, \beta, \gamma \in F$; put formally

$$u_1 = \sqrt{\alpha}\, e_1, \qquad u_2 = \sqrt{\beta}\, e_2, \qquad u_3 = \sqrt{\alpha\beta}\, e_3, \qquad u_4 = \sqrt{\gamma}\, e_4,$$

$$u_5 = \sqrt{\alpha\gamma}\, e_5, \qquad u_6 = \sqrt{\beta\gamma}\, e_6, \qquad u_7 = \sqrt{\alpha\beta\gamma}\, e_7;$$

then the multiplication table of the elements u_i, which is a consequence of table (1), contains only integral positive powers of the parameters α, β, γ:

$$u_1 u_2 = u_3, \qquad u_2 u_3 = \beta u_1, \qquad u_3 u_1 = \alpha u_2, \ldots, \tag{2}$$

and so on. Denote by $A(\alpha,\beta,\gamma)$ the 7-dimensional algebra over F with basis $\{u_i\}$ $(i = 1, \ldots, 7)$ and multiplication table (2). For any values of α, β, γ the algebra $A(\alpha,\beta,\gamma)$ is a Malcev algebra.

Theorem 1([11,12]). *In order that the algebra $A(\alpha,\beta,\gamma)$ be simple, it is necessary and sufficient that $\alpha, \beta, \gamma \neq 0$.*

We note that, when the condition of Theorem 1 is satisfied, the algebras $A(\alpha,\beta,\gamma)$ automatically turn out to be central simple; this follows from considerations of dimension. It is also clear that algebras $A(\alpha,\beta,\gamma)$ with different sets of parameters α, β, γ may be isomorphic; the question of isomorphism of these algebras will be resolved below (Theorem 3). In what follows F will denote an arbitrary field of characteristic 0, unless the contrary is stipulated, and all algebras under consideration will be assumed finite-dimensional.

It is well known (see, for example, ([2])) that the problem of classifying finite-dimensional simple algebras over F reduces to the problem of describing central simple algebras over F or over a finite extension of the field F.

Theorem 2. *Every central simple Malcev algebra over F is either a Lie algebra or is isomorphic to an algebra $A(\alpha,\beta,\gamma)$ for some $\alpha, \beta, \gamma \in F$.*

Proof. Let $A = A_F$ be a non-Lie central simple Malcev algebra over F. If the field F is algebraically closed, then the assertion is known ([6]). Suppose F is not algebraically closed and let Ω be the algebraic closure of F. Then the algebra $A_\Omega = A_F \otimes \Omega$ is central simple over Ω and, consequently, has the structure of the algebra $A(1,1,1)$ over Ω. Since the dimension of A_Ω over Ω is equal to the dimension of A_F over F, it follows from this, in particular, that the algebra A_F is 7-dimensional. We note some properties of the algebra $A(\alpha, \beta, \gamma)$ established in (^{12}).

a) On the linear space $A(\alpha, \beta, \gamma)$ there is defined a bilinear form (x, y) connected with multiplication in $A(\alpha, \beta, \gamma)$ by the formula

$$(xy)y = -(y, y)x + (x, y)y; \tag{3}$$

b) for any $x, y, z \in A(\alpha, \beta, \gamma)$,

$$(x, y) = (y, x), \qquad (xy, z) = (x, yz), \qquad (xy, xy) = (x, x)(y, y) - (x, y)^2; \tag{4}$$

c) if in $A(\alpha, \beta, \gamma)$ a basis $\{u_i\}$ with multiplication table (2) is chosen, and $x = \xi_1 u_1 + \dots + \xi_7 u_7$ $(\xi_i \in F)$, then

$$(x, x) = \alpha\xi_1^2 + \beta\xi_2^2 + \alpha\beta\xi_3^2 + \gamma\xi_4^2 + \alpha\gamma\xi_5^2 + \beta\gamma\xi_6^2 + \alpha\beta\gamma\xi_7^2. \tag{5}$$

Let $a_1, \dots, a_7$ be an arbitrary basis of A_F; these same elements may be taken as a basis of the algebra A_Ω. Then for the a_i the relations (3), (4) hold, where a priori $(a_i, a_j) \in \Omega$. Formula (3) shows that in fact $(a_i, a_j) \in F$. The simplicity of the algebra $A(\alpha, \beta, \gamma)$ is equivalent to the nondegeneracy of the bilinear form (x, y); consequently, the "Gram matrix" (a_i, a_j) of the basis $\{a_i\}$ of the algebra A_Ω must be nondegenerate. The established properties of the algebra A_F are sufficient to complete the proof. As u_1, u_2 choose arbitrarily two mutually orthogonal anisotropic elements of the algebra A_F; put $(u_1, u_1) = \alpha$, $(u_2, u_2) = \beta$; $\alpha, \beta \in F$, $\alpha\beta \neq 0$. Denote $u_3 = u_1 u_2$; then from (4) it follows that $(u_3, u_3) = \alpha\beta \neq 0$, $(u_3, u_1) = 0$, $(u_3, u_2) = 0$. Formula (3) shows that the elements u_1, u_2, u_3 form a basis of a three-dimensional subalgebra H in A_F with multiplication table

$$u_1 u_2 = u_3, \qquad u_2 u_3 = \beta u_1, \qquad u_3 u_1 = \alpha u_2.$$

The linear subspace H is nondegenerate with respect to the form (x, y); its orthogonal complement $H^\perp$ in A_F has the same property.

Choose in $H^\perp$ an arbitrary anisotropic element u_4 and set $(u_4, u_4) = \gamma$, $u_1 u_4 = u_5$, $u_2 u_4 = u_6$, $u_3 u_4 = -u_7$. As before, it is verified that the elements u_5, u_6, u_7 are anisotropic, and all elements u_i $(i = 1, \dots, 7)$ are pairwise orthogonal; consequently, they form a basis of A_F. With the help of formulas (3), (4), the multiplication table (2) is also recovered without difficulty.

The isomorphism problem for central simple Malcev algebras $A(\alpha, \beta, \gamma)$ over an arbitrary field F (possibly of finite characteristic) is solved by the following theorem.

Theorem 3. *Two algebras of type* $A(\alpha,\beta,\gamma)$ $(\alpha\beta\gamma \neq 0)$ *over one and the same field* F *of characteristic different from* 2 *are isomorphic if and only if the quadratic forms (5) defined for them are equivalent.*

The proof follows from the method of constructing the basis A_F described above, and from Witt’ s theorem.

In conclusion, let us consider the classification of finite-dimensional simple non-Lie Malcev algebras over the field of real numbers R. Let A be such an algebra, and suppose that A is not central. Then the centroid of A is isomorphic to the field of complex numbers C, and A may be regarded as central simple over C. The real dimension of A in this case is 14, and the multiplication table of A is uniquely determined. If A is central simple over R, then Theorems 2 and 3 are applicable. The criterion established by Theorem 3 shows that there exist altogether two nonisomorphic simple algebras of type $A(\alpha,\beta,\gamma)$ over R, corresponding to the values $(\alpha,\beta,\gamma) = (1,1,1)$ and $(1,1,-1)$, respectively. The latter result was noted earlier in ([12]). In that work there is also established a connection between the simple Malcev algebras $A(\alpha,\beta,\gamma)$ and the simple alternative algebras, i.e. the Cayley-Dickson algebras, which are characterized by the same parameters $\alpha,\beta,\gamma \in F$. Comparison of the known isomorphism criterion ([3]) for Cayley-Dickson algebras (of characteristic $\neq 2$) with Theorem 3 of the present paper shows that the algebras $A(\alpha,\beta,\gamma)$ are isomorphic if and only if the corresponding Cayley-Dickson algebras are isomorphic. Moreover, the automorphism groups of a Cayley-Dickson algebra and of the simple Malcev algebra obtained from it coincide.

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