ON THE NUMBER OF ELLIPTIC REGIONS ADJOINING A SINGULAR POINT

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Abstract

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Consider the autonomous system

$$\dot{x} = X_m(x,y) + \varphi(x,y) \equiv X(x,y), \quad \dot{y} = Y_m(x,y) + \psi(x,y) \equiv Y(x,y), \tag{1}$$

where X_m, Y_m are homogeneous polynomials of dimension $m \geq 1$, $X_m^2 + Y_m^2 \neq 0$, $\varphi, \psi = o(r^m)$, $r = \sqrt{x^2+y^2}$, and for which the origin O is an isolated singular point. We shall call system (1) a $C_{n,m}$-system if, in a neighborhood of the point O, $X, Y \in C_n$, and an A_m-system if X and Y are analytic at this point. I. Bendixson showed ([1]) that, for A_m-systems, the number e of elliptic regions adjoining the origin satisfies the inequality

$$e \leq 2m. \tag{2}$$

In the 1930s V. V. Stepanov proposed that this estimate could be replaced by the more precise one:

$$e \leq 2m - 2, \tag{3}$$

and V. V. Morozov came to the conclusion that inequality (3) is valid (see ([2])). However, it is easy to see that the singular point O of the A_m-system

$$\begin{aligned} \dot{x} &= [y + x^2][y + 3x^2] \dots [y + (2m-1)x^2], \\ \dot{y} &= xy[y + 2x^2] \dots [y + 2(m-1)x^2] \end{aligned} \tag{4}$$

is adjoined by $2m - 1$ elliptic regions; hence in fact inequality (3) is not fulfilled for any value of m. We shall show that in the analytic case estimate (2) admits a decrease by one, and we shall give some generalizations of this result to nonanalytic systems.

Lemma. *If the function $F(x,y)$ belongs to the class C_n in a neighborhood of the point (x_0, y_0), vanishes at the point (x_0, y_0) itself, and at least one of its n-th partial derivatives at this point is different from zero, then the branching index* ([3]) *of the curve $F = 0$ at the indicated point does not exceed $2n$.*

Proof. Without loss of generality one may assume that $F_{y^n}(x_0, y_0) \neq 0$. Arguing by contradiction, with successive application of Rolle' s theorem to $F, F_y, F_{y^2}, \dots, F_{y^{n-1}}$ as functions of y (with x fixed), we easily become convinced that, in a sufficiently small neighborhood of the point (x_0, y_0), every straight line parallel to the y-axis has no more than n common points with the curve $F = 0$, whence the validity of the lemma follows.

Theorem 1. *For A_m-systems the estimate*

$$e \leq 2m - 1, \tag{5}$$

holds, and it cannot be improved for any m.

Proof. In view of example (4), it is sufficient for us to establish inequality (5). At first we shall regard system (1) as a $C_{m,m}$-system.

Let $x = x(t),\ y = y(t)$ $(-\infty < t < +\infty)$ be the equations of one of its elliptic trajectories adjacent to the point O. Since $x, y, \dot{x}, \dot{y} \to 0$ as $t \to \pm\infty$, each of the functions $x(t), y(t)$ has at least one extremum and two points of inflection. It follows that any elliptic domain contains at least one O-branch (i.e., a branch entering the origin O) of each of the curves $X = 0$ and $Y = 0$, and no fewer than two O-branches of each of the curves

$$X\frac{\partial X}{\partial x} + Y\frac{\partial X}{\partial y} = 0, \qquad X\frac{\partial Y}{\partial x} + Y\frac{\partial Y}{\partial y} = 0. \tag{6}$$

By the lemma, the branching index at the origin of the curve $X = 0$ (or $Y = 0$) does not exceed $2m$, whence (2) follows. If, furthermore, the expression

$$\left[X_m\frac{\partial X_m}{\partial x} + Y_m\frac{\partial X_m}{\partial y}\right]^2 + \left[X_m\frac{\partial Y_m}{\partial x} + Y_m\frac{\partial Y_m}{\partial y}\right]^2 \tag{7}$$

does not vanish identically, and in a neighborhood of the origin $X, Y \in C_{2m}$, then the index at the point O of one of the curves (6) certainly does not exceed $4m - 2$, and the estimate (5) is valid. (Up to this point we have followed the line of reasoning of I. Bendixson and, in part, of B. V. Morozov.) Now suppose that the expression (7) vanishes identically and that system (1) is an A_m-system. Integrating system (6) in homogeneous polynomials of degree m, we obtain

$$X_m = cb(ax + by)^m, \qquad Y_m = -ca(ax + by)^m. \tag{8}$$

By a suitable linear transformation, system (1) is reduced to the form

$$\dot{x} = y^m + \varphi_1(x, y), \qquad \dot{y} = \psi_1(x, y), \tag{9}$$

where $\varphi_1, \psi_1 = o(r^m)$. We shall carry out the proof by contradiction. Suppose that $e = 2m$. Since through each elliptic domain there passes an O-branch of the curve $y^m + \varphi_1(x, y) = 0$, it is easy to conclude that points of the y-axis sufficiently close to the origin cannot belong either to elliptic or to parabolic domains. Consequently, system (9) has (true) hyperbolic domains ([4]).

On the other hand, since the curve $y^m + \varphi_1(x, y) = 0$ has $2m$ O-branches, then, by the Weierstrass preparation theorem and the known expansions of the roots of an algebraic equation with analytic coefficients ([5,6]), we rewrite system (9) in the form

$$\dot{x} = [y - \alpha_1(x)] \cdots [y - \alpha_m(x)][1 + \varphi_2(x, y)], \qquad \dot{y} = \psi_1(x, y),$$

where $\varphi_2(x, y)$ is an analytic function at the origin, $\varphi_2(0, 0) = 0$, and the functions $\alpha_k(x)$ are holomorphic in a neighborhood of the point $x = 0$, $\alpha_k(0) = \alpha_k'(0) = 0$, $k = 1, 2, \dots, m$. Introducing the transformation, regular at the point O, $x_1 = x$, $y_1 = y - \alpha_m(x)$, and returning to the old notation for the sought functions, we shall have

$$\dot{x} = y[y - \beta_1(x)] \cdots [y - \beta_{m-1}(x)][1 - \varphi_3(x, y)], \qquad \dot{y} = \psi_3(x, y),$$

where $\varphi_3(0, 0) = 0$, $\psi_3 = o(r^m)$, $\beta_k = \alpha_k - \alpha_m$, $k = 1, 2, \dots, m - 1$. All O-trajectories of the last system at the origin are still tangent to the x-axis. On this axis two elliptic domains are "strung." Setting $x = r\cos\vartheta$, $y = r\sin\vartheta$, we obtain

$$r\dot{r} = y[xy^{m-1} + \varphi_4(x, y)], \qquad r^2\dot{\vartheta} = -y^{m+1} + \psi_4(x, y),$$

where $\varphi_4 = o(r^m)$, $\psi_4 = o(r^{m+1})$. Each of the curves

$$y[xy^{m-1} + \varphi_4(x, y)] = 0, \tag{10^1}$$

$$-y^{m+1} + \psi_4(x, y) = 0 \tag{10^2}$$

has no more than $2m + 2$ O-branches. Consider an elliptic trajectory $r = r(t)$, $\vartheta = \vartheta(t)$. Obviously, $r, \vartheta \to 0$ as $t \to \pm\infty$, and for trajectories, ⋯

strung on the x-axis, $\vartheta(t)$, moreover, vanishes for the value t_* corresponding to the point of intersection of the trajectory with the x-axis. It follows that through each of the strung elliptic regions there passes a pair of O-branches of the curve (10^2) and one O-branch of the curve (10^1), while in each of the remaining elliptic regions there is one O-branch of these curves. All these branches are simple, and at the point O they are tangent to the x-axis. In addition, the curve (10^1) has two (simple) O-branches tangent at the origin to the y-axis. It is easy to show that if $y = y_1(x)$ and $y = y_2(x)$ are O-branches of the curves (10^1) and (10^2), belonging to one and the same elliptic region, then $|y_1(x)| < |y_2(x)|$ near and on that side of the point $x = 0$ where this inequality makes sense.

Enclose the point O in a rectangle of sufficiently small dimensions with sides parallel to the coordinate axes. From the preceding it follows that, when the contour of the rectangle is traversed in the direction of increasing ϑ, the points of its intersection with the O-branches of the curves (10^1) and (10^2) alternate. But in passing through the x-axis $dr/d\vartheta$ changes sign from plus to minus. Consequently, the same change of sign will occur in passing through each O-branch of the curve (10^1), which contradicts the previously established presence of hyperbolic regions. The theorem is proved.

In the course of the proof of the theorem the following proposition has been obtained:
For any $C_{m,m}$-system the inequality (2) is valid.

Remark. The following two systems

$$\begin{aligned} \dot{x} &= x^{2^{(k-1)+1/3^k}} (x^{2/3} - y^2)(x^{2/3} - y^2) \cdots (x^{2/3^k} - y^2), \\ \dot{y} &= y(x^2 - y^{14})(x^2 - y^{50}) \cdots (x^2 - y^{2(3^{k+1}-2)}), \end{aligned} \qquad (m = 2k - 1)$$

$$\begin{aligned} \dot{x} &= (x^{1/3^{k+1}} - y)\, x^{2k-1+2/3^{k+1}} (x^{2/3} - y^2) \cdots (x^{2/3^k} - y^2), \\ \dot{y} &= (x - y^{3^{k+2}-2})\, y(x^2 - y^{14}) \cdots (x^2 - y^{2(3^{k+1}-2)}), \end{aligned} \qquad (m = 2k)$$

$$(k = 1, 2, 3, \ldots),$$

for each of which $e = 2m+2$, show that for $C_{m-1,m}$-systems, for every natural m, the estimate (2) is false. Let us also note that if, for a normal two-dimensional autonomous system of class C_n, one assumes only the isolatedness of its singular point, then, in view of the equation*

$$\frac{dy}{dx} = \frac{x\left[x^{k-3}y^{k-1}(x^2 - y^2)\left(kxy\sin\frac{r^2}{2xy} - r^2\cos\frac{r^2}{2xy}\right)\sin\frac{r^2}{2xy} + r^{2k+2}\right]}{y\left[x^{k-1}y^{k-3}(x^2 - y^2)\left(kxy\sin\frac{r^2}{2xy} - r^2\cos\frac{r^2}{2xy}\right)\sin\frac{r^2}{2xy} - r^{2k+2}\right]},$$

which admits the integral

$$r^{2k+2} = 2x^k y^k \sin^2\frac{r^2}{2xy} + C,$$

an even number of elliptic regions may adjoin the singular point. Analogously to Theorem 1, one proves

Theorem 2. *For $C_{3,1}$-systems the unimprovable estimate $e \leqslant 1$ is valid.*

Instead of the considerations connected with the preparatory theorem of Weierstrass, one should use the theorem on the existence of an implicit function (of class C_3), and, in estimating the indices of the curves (10), the lemma. Let us note that for A_1-systems the estimate $e \leqslant 1$ follows from ([8],[9]).

Theorem 3. *Let e and h be respectively the numbers of elliptic and hyperbolic regions of a $C_{m+1,m}$-system adjoining the origin. Then*

$$e + h \leqslant 2m + 2, \tag{11}$$

and the estimate $h \leqslant 2m + 2$, and a fortiori (11), cannot be improved.

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* The principle of constructing similar examples is set out in (7).

Proof. Let K be a closed disk of sufficiently small radius ρ with center at the point O, and let $x = x(t)$, $y = y(t)$ ($t_1 \leq t \leq t_2$) be an elliptic trajectory ($t_1 = -\infty$, $t_2 = +\infty$), or a maximal connected piece, contained in K, of a hyperbolic trajectory (t_1, t_2 finite). Since in each of these cases, at the ends of the segment $[t_1, t_2]$, the function $x^2 + y^2$ assumes equal values, there is a point $t_0 \in (t_1, t_2)$ at which $xX + yY = 0$. Consequently, through every elliptic and hyperbolic domain there passes at least one O-branch of this curve. By stretching one of the coordinate axes one can always arrange that $xX_m + yY_m \neq 0$, so that the branching index of the indicated curve at the point O does not exceed $2m + 2$, whence (11) follows. Changing the sign of the right-hand side of one of equations (4) to the opposite one, we obtain a system for which $h = 2m + 2$, which completely proves the theorem.

Theorem 4. *If X_m and Y_m cannot be represented simultaneously in the form (8), then for the $C_{m+1,m}$-system (1) the estimate (5) holds, and for the $C_{3m,m}$-system (1) the estimate (3) holds, and it cannot be improved for any m.*

Proof. Let system (1) be a $C_{m+1,m}$-system. From Bendixson' s formula[6] for the index j of a singular point,

$$e - h \leq 2(j - 1),$$

and from the inequality $|j| \leq m$, which follows immediately from the definition of the Poincaré index, we obtain

$$e - h \leq 2(m - 1). \tag{12}$$

Suppose that $e = 2m$. From (11) and (12) it follows that the system has two hyperbolic domains. Assume first that the angle between the separatrices of each of these domains is equal to π. Then all O-trajectories at the origin will be tangent to one straight line. Let us make it coincide with the x-axis. It is easy to see that system (1) will take the form (9). And since the representability of X_m and Y_m in the form (8) (or, what is the same, the identical vanishing of expression (7)) is a property preserved under nonsingular linear transformations, we have obtained a contradiction with the condition of the theorem. Let us consider the second possibility: the opening angle of at least one of the hyperbolic domains is less than π. Using Bendixson' s theorem on continuation of a characteristic through a singular point[1] and the continuity of the field of directions near an ordinary point, we conclude that by rotating the coordinate system through a suitable angle one can arrange that $Y_m \neq 0$, and that at least one O-branch of the curve $Y = 0$ passes through one of the hyperbolic domains. According to the preceding, in each of the $2m$ elliptic domains there is also at least one O-branch of this curve, which is impossible by the lemma. The first part of the theorem is proved. The second part follows from [2] with allowance for the lemma.

From consideration of the family of "roses" $r = C\sin(m-1)\vartheta$ and its differential equation, and from inequality (12), there follows the proposition:

If for a $C_{m+1,m}$-system $h = 0$, then the unimprovable estimate (3) is valid for it.

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Note: Figure translations are in progress. See original paper for figures.