ON ONE ANALOGUE OF THE HARDY-LITTLEWOOD EQUATION

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Abstract

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ON ONE ANALOGUE OF THE HARDY–LITTLEWOOD EQUATION

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In the monograph of Yu. V. Linnik ([1]), among analogues of the Hardy–Littlewood equation, the equation

$$p_1 p_2 + \xi^2 + \eta^2 = n, \tag{1}$$

is considered, where p_1 and p_2 independently run through the prime numbers; ξ and η run through the integers; n is a sufficiently large natural number (the main parameter).

For the number $S(n)$ of solutions of equation (1), in ([1]) an asymptotic formula is derived under the additional restrictions

$$p_1,\ p_2 > \exp(\ln\ln n)^2.$$

The aim of the present note is to remove these restrictions and to obtain $S(n)$ with a better remainder term than in ([1]).

Theorem. *As $n \to \infty$,*

$$S(n) = \pi A_0 A(n) \frac{n}{\ln n} \prod_{p|n} \frac{(p-1)(p-\chi(p))}{p^2 - p + \chi(p)} + O\left(\frac{n}{(\ln n)^{1.042}}\right), \tag{2}$$

where

$A_0 = \prod_{p>2} \left(1 + \frac{\chi(p)}{p(p-1)}\right)$; $\chi(m)$ is the nonprincipal character mod 4; $A(n)$ is

some arithmetical factor, whose structure is clear from the proof.

We precede the proof of the theorem with two lemmas. Consider the equation

$$ap + b(\xi^2 + \eta^2) = n, \tag{3}$$

where a, b are integers satisfying the conditions: $a = O(\exp(\ln n)^{\alpha})$, $b = O(\ln^C n)$, $C > 0$ is a sufficiently large constant, and p runs through the prime numbers. We shall assume that $(a,b) = 1$, $(ab, 2n) = 1$. The latter conditions are not essential.

Let $Q(n)$ be the number of solutions of equation (3).

Lemma 1. *As $n \to \infty$,*

$$Q(n) = \pi A_0 \frac{n}{ab \ln n} \prod_{p|an} \frac{(p-1)(p-\chi(p))}{p^2 - p + \chi(p)} \prod_{p|b} \frac{p^2}{p^2 - p + \chi(p)} + O\left(\frac{n}{ab(\ln n)^{1.042}}\right). \tag{4}$$

For the proof of the lemma, following C. Hooley ([4]) and B. M. Bredikhin ([4]), we represent $Q(n)$ in the following form:

$$\begin{aligned} Q(n) &= \sum_{ap+b(\xi^2+\eta^2)=n} 1 = 4 \sum_{ap+2^\lambda bxy=n} \chi(x) = 8 \sum_{\substack{ap+2^\lambda bxy=n \\ x \le \sqrt{n}\, n_1^{-1}}} \chi(x) \\ &\quad - 4 \sum_{\substack{ap+2^\lambda bxy=n \\ \sqrt{n}\, n_1^{-1} < x < \sqrt{n}\, n_1 \\ y < \sqrt{n}\, n_1}} \chi(x) + 4 \sum_{\substack{ap+2^\lambda bxy=n \\ \sqrt{n}\, n_1^{-1} < x < \sqrt{n}\, n_1}} \chi(x) + O\left(\frac{n}{ab \ln^2 n}\right) \\ &= \Sigma_A - \Sigma_{B_1} + \Sigma_{B_2} + O\left(\frac{n}{ab \ln^2 n}\right), \end{aligned}$$

where $n_1 = \exp(\ln n)^{\alpha_1}$; $\alpha_0, \alpha_1 > 0$ are sufficiently small constants.

Σ_A is evaluated with the aid of E. Bombieri' s lemma ([5]), while Σ_{B_1} and Σ_{B_2} are estimated by S. Hooley' s method ([4]) according to the scheme developed in ([8]).

Equation (3) may be regarded as a special case of the linear equation $ax + by = n$, when x runs through the primes and y assumes values of the quadratic form $\varphi(\xi, \eta) = \xi^2 + \eta^2$. This equation is a natural generalization of the classical Hardy-Littlewood equation and is of interest from the point of view of possible applications (see, for example, below and ([7],[8])).

Next, let $S_1(n)$ be the number of solutions of equation (1) when

$$p_1,\ p_2 > P_0 = \exp(\ln n)^{\alpha_0}.$$

Lemma 2. As $n \to \infty$,

$$S_1(n) = \pi A_0 \prod_{p|n} \frac{(p-1)(p-\chi(p))}{p^2-p+\chi(p)} \sum_{\substack{p_1p_2<n \\ p_1p_2>P_0}} 1 + O\left(\frac{n}{(\ln n)^{1.042}}\right). \quad (5)$$

Proof. We write $S_1(n)$ in the form

$$S_1(n) = 2 \sum_{\substack{p_1p_2+\xi^2+\eta^2=n \\ P_0<p_1<P}} 1 + \sum_{p_1p_2+\xi^2+\eta^2=n} 1 + O\left(\frac{n}{\ln^2 n}\right) =$$

$$= 2S_2(n) + S_3(n) + O\left(\frac{n}{\ln^2 n}\right), \quad (6)$$

where $P = \exp \ln n \dfrac{\ln\ln\ln n}{K \ln\ln n}$, and K is a sufficiently large constant.

With the aid of the dispersion method ([1]) (see ([6],[13],[10]) and p. 131) we obtain

$$S_2(n) = \pi A_0 \prod_{p|n} \frac{(p-1)(p-\chi(p))}{p^2-p+\chi(p)} \sum_{\substack{p_1p_2<n \\ P_0<p_1<P}} 1 + O\left(\frac{n}{\ln^2 n}\right). \quad (7)$$

Refining the proof of Theorem 2 of ([2]), we obtain

$$S_3(n) = \pi A_0 \prod_{p|n} \frac{(p-1)(p-\chi(p))}{p^2-p+\chi(p)} \sum_{\substack{p_1p_2<n \\ p_1,\ p_2>P}} 1 + O\left(\frac{n}{(\ln n)^{1.042}}\right). \quad (8)$$

From (6)–(8), (5) follows.

Proof of the theorem. Decompose $S(n)$ as follows:

$$S(n) = 2 \sum_{\substack{p_1p_2+\xi^2+\eta^2=n \\ p_1<P_0}} 1 + S_1(n) + O\left(\frac{n}{\ln^2 n}\right) = 2S_4(n) + S_1(n) + O\left(\frac{n}{\ln^2 n}\right). \quad (9)$$

We find $S_4(n)$:

$$S_4(n) = \sum_{\substack{(p_1,n)=1 \\ p_1<P_0}} \sum_{p_1p_2+\xi^2+\eta^2=n} 1 + \sum_{\substack{p_1|n \\ p_1<P_0}} \sum_{p_1p_2+\xi^2+\eta^2=n} 1 = \Sigma_1 + \Sigma_2. \quad (10)$$

The inner sum in Σ_1 is evaluated with the aid of Lemma 1. We obtain

$$\Sigma_1 = \sum_{\substack{(p_1,n)=1 \\ p_1<P_0}} \left(\pi A_0 \frac{n}{p_1 \ln n} \prod_{p|p_1 n} \frac{(p-1)(p-\chi(p))}{p^2-p+\chi(p)} + O\left(\frac{n}{p_1 (\ln n)^{1.042}}\right)\right). \quad (11)$$

Further, by virtue of

$$\sum_{\substack{d|ml \\ (m,l)=1}} \chi(d) = \sum_{d|m} \chi(d) \sum_{d|l} \chi(d)$$

and Lemma 1, we have (with ad-

with admissible error)

$$\Sigma_2 = 4 \sum_{\substack{p_1|n \\ p_1<P_0}} \sum_{\lambda,\ s=0,1\dots} \sum_{d|p_1^{1+s}} \chi(d) \sum_{p_2+2^\lambda p_1^{sxy}=n_0} \chi(x)$$

$$= \sum_{\substack{p_1|n \\ p_1<P_0}} \sum_{s=0,1,\dots} \sum_{d|p_1^{1+s}} \chi(d) \left(\pi A_0 \frac{n}{p_1^{1+s} \ln n} \prod_{p|n_0} \frac{(p-1)(p-\chi(p))}{p^2-p+\chi(p)} \frac{p_1^2}{p_1^2-p_1+\chi(p_1)} + \right.$$

$$\left. + O\left(\frac{n}{p_1^{1+s} (\ln n)^{1.042}}\right)\right), \quad (12)$$

where $n_0 = n/p_1$.

Now (2) follows from (5), (9)-(12).

By the methods of the papers ([6],[8],[9]) one can derive geometric and ergodic properties of the solutions of equation (1).

Equation (1) can be generalized. By applying analogous means one can treat the equation

$$p_1 p_2 \dots p_k + \varphi(\xi, \eta) = n,$$

where $k > 2$, and $\varphi(\xi, \eta)$ is some prescribed quadratic form.

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Note: Figure translations are in progress. See original paper for figures.