# Reports of the Academy of Sciences of the USSR

MATHEMATICS

1968

SovietRxiv

## Abstract

## Full Text

Reports of the Academy of Sciences of the USSR
1968. Volume 181, No. 3

UDC 512.25/.26+519.3:330.115

*MATHEMATICS*

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# ON THE QUESTION OF DETERMINING THE TOTALITY OF ALL OPTIMAL SOLUTIONS OF A LINEAR PROGRAMMING PROBLEM

*(Presented by Academician A. N. Kolmogorov on 13 XI 1967)*

In a communication ([1]) by the first of the authors, an extremely simple derivation was noted of the general solution (i.e., the set $K^*$ of all optimal solutions) of a linear programming (l.p.) problem from a parametric representation of the convex polyhedron $K$ of "admissible solutions" (plans). There, too, the potential practical importance was noted of the $K^*$-problem (the effective complete determination of the set $K^*$), to which too little attention had been paid in the literature; the transition proposed in that same article from the parametric representation of $K$ to the corresponding representation of $K^*$ was considered primarily in its theoretical—fundamental—aspect.

In a recent work of the authors ([2]), which in its immediate orientation was aimed at a somewhat different range of diverse applications, a comparatively simple general computational method was developed, characterized as a branched autonomous-simplex (without objective function) process (the b.a.s.-method), for determining all vertices of an arbitrarily given convex polyhedron (corresponding to a system of linear inequalities) (problem V). To this problem V, or else to the successive solution of two such problems, is reduced the solution of problem Π—finding a parametric representation of the set of points of any convex nonprismatic polyhedron (respectively bounded or unbounded). Applying, in particular, this b.a.s.-method to the computational solution of problem Π for the polyhedron $K$ of admissible solutions of an l.p. problem, from its parametric representation we can almost immediately obtain, according to the results of ([1]), also a parametric determination of the sought set $K^*$ of optimal solutions of the l.p. problem itself.

However, a closer examination of the latter question from the standpoint of practical implementation, with special allowance for the fact of a very substantial increase in the labor required to solve problems V as the dimension and the number of vertices of convex polyhedra increase, leads to the conclusion that there is an undoubted advantage in a more direct solution, insofar as possible, of problem Π (with the aid of the same b.a.s.-method) for the sought polyhedron $K^*$ itself, which constitutes only a structural part of the polyhedron $K$. The purpose of the present communication is to show that precisely such an approach becomes not only possible but also quite feasible if one proceeds from the natural setup: to pose the question of the numerical determination of the polyhedron $K^*$ of all optimal solutions of an l.p. problem after solving (for example, by the ordinary simplex method) the primary problem of finding ([4]) one of the optimal solutions.

Let, for definiteness, a canonical maximum problem of l.p. be considered, which, with the explicit form of the constraints (using below the "compressed ticker" ([3]) notation of simplex tableaus) and under the assumption of an already found admissible basic solution, can be written as follows.

formulated as:

$$w = -\sum_{\nu=1}^{n} g_\nu u_{i_\nu} + \tau = \max! \tag{1}$$

under the conditions

$$u_{i_{n+s}} = -\sum_{\nu=1}^{n} \alpha_{s\nu} u_{i_\nu} + \gamma_s \quad (s = 1, \dots, m-n;\ \gamma_s \geq 0), \tag{2}$$

$$u_i \geq 0 \quad (i = 1, \dots, m). \tag{3}$$

Suppose that, as a result of carrying out the ordinary simplex process (its main phase), an optimal basic solution has been obtained, represented in the simplex table, which, with a possible change in the numbering of the variables, may be written in the form (4), where, for example, the last row means:

$$w = -\sum_{\nu=1}^{n} h_\nu u_\nu + \tau^{(0)}$$

($\tau^{(0)}$ is the sought maximum value of the objective function $w$).

| | $-u_1$ | $\cdots$ | $-u_n$ | $1$ |
|---|---|---|---|---|
| $u_{n+1} =$ | $a_{11}$ | | $a_{1n}$ | $d_1$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ |
| $u_m =$ | $a_{m-n,1}$ | $\cdots$ | $a_{m-n,n}$ | $d_{m-n}$ |
| $w =$ | $h_1$ | $\cdots$ | $h_n$ | $\tau^{(0)}$ |

(4)

$$h_\nu \geq 0, \quad d_s \geq 0 \quad (\nu = 1, \dots, n;\ s = 1, \dots, m-n).$$

If there exist other optimal solutions $u = (u_1, \dots, u_m)$, distinct from the one found

$$u = (0_1, \dots, 0_n, d_1, \dots, d_{m-n}),$$

then they must, obviously, satisfy the relation

$$-h_1 u_1 - \dots - h_{nu} n = 0, \tag{5}$$

which also characterizes all possible optimal solutions $(u_1, \dots, \dots, u_m)$—with observance, of course, also of the $(m-n)$ relations

$$u_{n+s} = -\sum_{\nu=1}^{n} a_{s\nu} u_\nu + d_s$$

together with the inequalities (3).

Taking into account the nonnegativity of all $n$ coefficients $h_\nu$, we immediately see from (5) that the existence of optimal solutions different from the one found is possible only in the presence of degeneracies of the second kind, i.e., when some $h_\nu$ vanish. Allowing, for generality, also the possible presence of degeneracies of the first kind (with which the remark at the end of the article concerning the dual L.P. problem is connected), let us specifically have (with a possible, again, partial change in the numbering of the variables):

$$\begin{aligned} h_1 = h_2 = \dots = h_k = 0 \quad (1 \le k < n)^*, \\ d_1 = d_2 = \dots = d_l = 0 \quad (0 \le l < m-n). \end{aligned} \tag{6}$$

In this case (5) is equivalent to the collection of $(n-k)$ conditions

$$u_{k+1} = u_{k+2} = \dots = u_n = 0. \tag{7}$$

The matter reduces to determining the set of optimal "reduced"($(m-n+k)$-term) tuples

$$u' = (u_1, \dots, u_k;\ u_{n+1}, \dots, u_m).$$

This set coincides exactly with the set of all points of a convex

* To estimate the degree of economy in introducing, say, table (8) instead of (4) as a basis for the process of c

of the nonprismatic polyhedron $S \equiv K'^*$ of the space $R_{m-n+k} = \{u'\}$, given by the $m-n$ relations of table (8) together with the conditions (9) of nonnegativity of all $m-n+k$ components of the selection $u'$:

$$
\begin{array}{r|ccc|c}
 & -u_1 & \cdots & -u_k & 1 \\
\hline
u_{n+1} = & a_{11} & \cdots & a_{1k} & d_1 \\
\vdots & \vdots & & \vdots & \vdots \\
u_m = & a_{m-n,1} & \cdots & a_{m-n,k} & d_{m-n}
\end{array}
\tag{8}
$$

$$
u_1 \geqslant 0, \dots, u_k \geqslant 0; \qquad u_{n+1} \geqslant 0, \dots, u_m \geqslant 0. \tag{9}
$$

Solving, by the r.a.s. method ([2]), problem V for this polyhedron $S$, we find all its vertices ${u'}^{(1)}, \dots, {u'}^{(p)}$ $(p \geqslant 1)$. But for the complete determination of the set ${K'}^* \equiv S$ (possibly unbounded!) it is also necessary to solve problem V for the auxiliary convex polyhedron $P$ (the cross-section of the potential cone of directing rays), which, in the $(m-n+k)$-dimensional space of points $\dot{u}' = (\dot{u}_1, \dots, \dot{u}_k; \dot{u}_{n+1}, \dots, \dot{u}_m)$, is defined by the $m-n+1$ relations of table (10) together with the $m-n+k$ conditions $\dot{u}_j \geqslant 0$.

$$
\begin{array}{r|ccc|c}
 & -\dot{u}_1 & \cdots & -\dot{u}_k & 1 \\
\hline
\dot{u}_{n+1} = & a_{11} & \cdots & a_{1k} & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
\dot{u}_m = & a_{m-n,1} & \cdots & a_{m-n,k} & 0 \\
0 = & 1 & \cdots & 1 & 1
\end{array}
\tag{10}
$$

Having performed in table (10) a Jordan substitution of the type $(0 \dots \dot{u}_r)$, $r \in \{1, \dots, k\}$, as a result of which the left column takes the form $\langle \dot{u}_{n+1}, \dots, \dot{u}_m, \dot{u}_r \rangle$, and omitting the column with the heading $-0$, which should have appeared, we obtain a table of a form similar to (8), but with a numerical matrix of dimensions $(m-n+1) \times (\overline{k-1}+1)$, and among the elements of the right-hand column there may also be negative ones.* In the latter case we use the well-known trial procedure for eliminating such elements (cf., for example, ([4]), p. 1266) and, if its outcome is positive, then apply the same r.a.s. method ([2]), which will also furnish us with the totality of all vertices $\dot{u'}^{(1)}, \dots, \dot{u'}^{(q)}$ of the polyhedron $P$, or, what is the same thing, the totality of the vectors of extreme rays of the convex polyhedron $S \equiv {K'}^*$.

The required set $K^*$ of all optimal solutions of our l.p. problem (1)–(3) is finally determined parametrically in the form

$$
u = \sum_{i=1}^{p} \lambda_i u^{(i)} + \sum_{j=1}^{q} \mu_j \dot{u}^{(j)} \qquad \left( \lambda_i \geqslant 0,\ \sum_{i=1}^{p} \lambda_i = 1;\ \mu_j \geqslant 0 \right), \tag{11}
$$

where $u^{(i)}, \dot{u}^{(j)}$ are obtained from ${u'}^{(i)}, \dot{u'}^{(j)}$ by supplementing these reduced sets with the insertion of $n-k$ zero components with numbers $k+1, \dots, n$; here in general $p \geqslant 1$, $q \geqslant 0$.

Let us add the following remark concerning the l.p. problem dual to (1)–(3). Performing in table (4) a replacement of headings, namely: $v_{n+s}$ instead of $u_{n+s} =$ $(s = 1, \dots, m-n)$, $v_v =$ instead of $-u_v$ $(v = 1, \dots, n)$; 1 instead of $w =$, and $\widetilde{w} =$ instead of 1, we immediately have, for the dual minimum problem, from the numerical matrix of table (4)

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* This is exactly when, in table (10), above the key element 1 of the column $-u_v$, there are some positive elements.

optimal basic solution $v = (h_1, \dots, h_n, 0_{n+1}, \dots, 0_m)$, with the same optimal value of the objective function $\widetilde{w}(= \widetilde{w}_{\min}) = \tau^{(0)}$. Starting from the very same table (4), with changed headings, we can, for $l > 0$ ($l$ is the number of the above-mentioned degeneracies of the first kind—see (6)), determine also all sets $\widetilde{K}^*$ of optimal solutions of the dual linear-programming problem. For this purpose, now in the table $(\tilde{4})$, while retaining the lower row $(1;\ h_1, \dots, h_n)$ (without $\tau^{(0)}$), we omit the right column $\langle \widetilde{w};\ d_1, \dots, d_{m-n};\ \tau^{(0)} \rangle$ and $m-n-l$ rows $(v_{n+s};\ a_{s1}, \dots, a_{sn})$ $(s = l+1, \dots, m-n)$; then we proceed in natural analogy with the preceding, applying successively twice (with resultant indicators $\tilde{p} \geqslant 1$, $\tilde{q} \geqslant 0$) the same method of r.a.s. [2] with obvious technical modifications (in particular, the rule of the negative key element replaces the former rule of the positive one).

It should also be noted that, under such a joint consideration of a pair of mutually dual linear-programming problems, the equalities (7), which played an essential role in formulating the economical method for determining the set $K^*$, turn out to be nothing other than a concrete manifestation of the general principle ((5), p. 180) of "complementary slackness."

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Received
5 XI 1967

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*Note: Figure translations are in progress. See original paper for figures.*