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Abstract

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NECESSARY AND SUFFICIENT CONDITIONS FOR THE STABILITY OF TWO-LEVEL DIFFERENCE SCHEMES

In papers ([1–3]), sufficient conditions for stability were found and a priori estimates were obtained for two-level and three-level schemes with variable (in t) operators, specified in an abstract real Hilbert space.

In the present paper, two-level schemes ([2]) are considered. It is shown that for schemes with constant operators A and $\tilde{B}$ the necessary and sufficient conditions coincide. These same conditions are sufficient for stability in the class of schemes with variable operators $A(t)$ and $B(t)$. The closeness of the sufficient and necessary conditions for schemes with variable operators makes it possible to judge a theorem on a sufficient condition for instability.

The method of investigating stability, unlike ([1], [3]), is based on reducing a scheme of general form to an explicit scheme and subsequently estimating the transition operator for the explicit scheme. Here we restrict ourselves to studying stability with respect to the initial data. For references to works on the stability of difference schemes, see ([3]).

1. Let $\{H_h\}$ be a family of real Hilbert spaces depending on a parameter h, which is a vector of some normed space (for example, Euclidean space R_N of dimension N, cf. ([1–3])); $|h|$ is the norm of the vector h. Consider, on the interval $0 \le t \le t_0$, the mesh $\omega_\tau = \{t_n = n\tau,\ n = 0, 1, \dots, n_0,\ \tau = t_0/n_0\}$.

Let further $y(t) = y_{h\tau}(t)$, $\varphi(t) = \varphi_{h\tau}(t)$, etc., be abstract functions of the argument $t \in \omega_\tau$ with values in H_h; $A(t) = A_{h\tau}(t)$, $B(t) = B_{h\tau}(t)$, $C(t) = C_{h\tau}(t)$, etc., are linear operators mapping H_h into H_h for each $t \in \omega_\tau$. The first-order difference equation with operator coefficients

$$B_{h\tau}(t) y_{h\tau}(t+\tau) = C_{h\tau}(t) y_{h\tau}(t) + \tau\varphi_{h\tau}(t), \quad 0 \le t = n\tau < t_0,$$
$$y_{h\tau}(0) = y_{0h\tau} \in H_h, \tag{1}$$

where $\varphi_{h\tau}(t)$ is a given function, will be called a two-level scheme ([2]). To simplify the notation, the indices h, τ will henceforth be omitted.

Any two-level scheme can be written in canonical form

$$B(t)\frac{y(t+\tau)-y(t)}{\tau}+A(t)y(t)=\varphi(t),\quad 0\le t=n\tau<t_0,\quad y(0)=y_0\in H. \tag{2}$$

Let (,) and $\|y\|=\sqrt{(y,y)}$ be the scalar product and norm in H. We shall write $A=A^*>0$ if A is a self-adjoint and positive ($(Ax,x)>0$ for all $x\in H$ with $\|x\|\ne 0$) operator; $A\ge B$ if $(Ax,x)\ge(Bx,x)$ for all $x\in H$. Along with H we shall consider the energy spaces H_A and H_B, consisting of the same elements as H, with norms $\|y\|_A=\sqrt{(Ay,y)}$ in H_A ($A=A^*>0$); $\|y\|_B=\sqrt{(By,y)}$ in H_B ($B=B^*>0$).

We consider a real Hilbert space in order to take into account the case of non-self-adjoint positive operators.

2. In this paper we study only stability with respect to the initial data (with respect to i.d.). Therefore let us consider the homogeneous equation (2) with $\varphi=0$:

$$B(t)\frac{y(t+\tau)-y(t)}{\tau}+A(t)y(t)=0,\qquad 0\le t=n\tau<t_0,$$
$$\text{given } y(0)=y_0\in H. \tag{3}$$

We shall say that scheme (2) is stable with respect to i.d. if one can specify a real constant c_0, independent of h and τ, such that, for sufficiently small $|h|\le h_0$ and $\tau\le\tau_0$, for the solution of problem (3) with arbitrary $y(0)=y_0\in H$ the estimate

$$\|y(t)\|_{(1)}\le e^{c_0t}\|y(0)\|_{(1_0)}\quad \text{for all } t\in\omega_\tau \tag{4}$$

holds, where $\|\cdot\|_{(1)}$ and $\|\cdot\|_{(1_0)}$ are certain norms on the set H (cf. (1^{-4})).

Scheme (3) is absolutely stable if it is stable for all $\tau>0$ and $|h|>0$. In all the theorems formulated below, scheme (3) is absolutely stable if the sufficient conditions are satisfied for all $\tau>0$ and $|h|>0$. We shall say that: 1) scheme (3) is stable in H_A, if (4) is fulfilled and $\|\cdot\|_{(1)}=\|\cdot\|_{(1_0)}=\|\cdot\|_A$, where A does not depend on t; 2) scheme (3) is stable in $H_{A(t)}$, if $\|y(t+\tau)\|_{A(t)}\le e^{c_0(t+\tau)}\|y(0)\|_{A(0)}$. Stability in H_B and $H_{B(t)}$ is understood analogously.

3. Consider the explicit scheme

$$x(t+\tau)=Sx(t),\quad S=E-\tau C,\quad 0\le t=n\tau<t_0,\quad x(0)=x_0\in H \tag{5}$$

with transition operator S; here E is the identity operator, and x_0 is an arbitrary vector. If one of the operators A or B is self-adjoint, positive, and constant, then (3) reduces to an explicit scheme with operators

$$C_1 = A^{1/2} B^{-1} A^{1/2} \quad \text{or} \quad C_2 = B^{-1/2} A B^{-1/2}.$$

Lemma 1. *Let $A = A^* > 0$ be independent of t, and let $B^{-1}(t)$ exist. Then (3) and (5) are equivalent for $C = C_1,\ x = A^{1/2} y$. If $B = B^* > 0$ is a constant operator, then (3) and (5) are equivalent for $C = C_2,\ x(t) = B^{1/2} y(t)$, or for $C = C_2,\ x(t) = B^{-1/2} A y(t)$ (if A is also constant).*

Indeed, let $A = A^* > 0$. Then there exists $A^{1/2} = (A^{1/2})^* > 0$ (see ([5])). Applying the operator $A^{1/2} B^{-1}$ to (3), we obtain, for $x(t) = A^{1/2} y(t)$, if A is constant, scheme (5) with $C = C_1$, so that $\|x(t)\| = \|y(t)\|_A$, and so on.

Lemma 1 allows one to reduce the study of stability of scheme (3) in H_A or H_B to the study of stability of the explicit scheme (5) in H:

$$\|x(t)\| \le e^{c_0 t} \|x(0)\|. \tag{6}$$

4. We shall need the definition of the norm of an operator S in H: $\|S\| = \sup_{\|x\|=1} \|Sx\|$, and the equivalent definition for $S = S^*$ ([5]),

$$\|S\| = \sup_{\|x\|=1} |(Sx, x)|, \tag{7}$$

as well as a number of lemmas, valid also for operators depending on t (under the assumption that all conditions are fulfilled for each $t = n\tau \in [0, t_0)$).

Lemma 2. *If $C = C^*$, then the condition*

$$\frac{1-\rho}{\tau} E \le C \le \frac{1+\rho}{\tau} E \quad \text{or} \quad \frac{1-\rho}{\tau} \|x\|^2 \le (Cx, x) \le \frac{1+\rho}{\tau} \|x\|^2, \tag{8}$$

where $\rho > 0$, is necessary and sufficient for the estimate

$$\|S\| \le \rho, \qquad S = E - \tau C \tag{9}$$

(conditions (8) and (9) are equivalent).

Let (8) be fulfilled, i.e. $-\rho E \le \tau C - E \le \rho E$. Hence, from (7), it follows that $\|S\| = \|-S\| \le \rho$. The converse course of reasoning is obvious.

Lemma 3. *If $C = C^* > 0$, then the inequalities $\gamma_1 E \le C \le \gamma_2 E$ and*

$$\frac{1}{\gamma_2} E \le C^{-1} \le \frac{1}{\gamma_1} E$$

are equivalent.

Lemma 4. If $C = C^* > 0$, then conditions (9) and

$$C^{-1} \geqslant \frac{\tau}{1+\rho} E, \qquad \rho > 0 \tag{10}$$

are equivalent for $\rho \geqslant 1$.

Let $C > 0$ be a non-self-adjoint operator. Then (10) is sufficient for $\rho \geqslant 1$, necessary for $\rho \leqslant 1$, and necessary and sufficient for $\rho = 1$ for the estimate (9).

Lemma 4 for $C = C^*$ follows from Lemmas 3 and 2. Let $C = C^*$ and suppose (10) is satisfied. Since $(1+\rho)(C^{-1}x, x) - \tau\|x\|^2 = (1+\rho)(Cy, y) - \tau\|Cy\|^2$, where $y = C^{-1}x$, it follows from (10) that $\tau\|Cy\|^2 \leqslant (1+\rho)(Cy, y)$, $\tau(Cy, y) \leqslant (1+\rho)\|y\|^2$. Therefore

$$\|Sy\|^2 = \|(E-\tau C)y\|^2 = \|y\|^2 - 2\tau(Cy, y) + \tau^2\|Cy\|^2 \leqslant \|y\|^2 + \tau(\rho-1)(Cy, y) \leqslant \rho^2\|y\|^2$$

for $\rho \geqslant 1$, i.e. $\|S\| \leqslant \rho$. If $\rho = 1$, then from $\|Sy\|^2 \leqslant \|y\|^2$ it follows at once that $0.5\tau\|Cy\|^2 \leqslant (Cy, y)$, or $C^{-1} \geqslant 0.5\tau E$.

Lemma 5. The inequality

$$C^{-1} \geqslant \gamma E, \qquad \gamma > 0,$$

is equivalent to one of the inequalities: 1) $B \geqslant \gamma A$ for $A = A^* > 0$, $B > 0$, $C = C_1 = A^{1/2}B^{-1}A^{1/2}$, or for $B = B^* > 0$, $A = A^* > 0$, $C = C_2 = B^{-1/2}AB^{-1/2}$; 2) $A^{-1} \geqslant \gamma B^{-1}$ for $B = B^* > 0$, $A > 0$, $C = C_2$.

Lemma 6. If $C = C_1$, $B = B^* > 0$, $A = A^* > 0$, or $C = C_2$, $B = B^* > 0$, $A > 0$, then the inequalities $\gamma_1 E \leqslant C \leqslant \gamma_2 E$ and $\gamma_1 B \leqslant A \leqslant \gamma_2 B$, for $\gamma_1 > 0$, $\gamma_2 > 0$, are equivalent.

5. For the case of constant A and B we shall find coincident necessary and sufficient conditions for stability of the scheme (3) in H_A and H_B. Rewrite (5) in the form

$$x_{n+1} = Sx_n, \quad \text{where } S = E - \tau C, \quad x_n = x(n\tau), \quad 0 \leqslant n < n_0, \quad x_0 \in H \text{ is given.} \tag{11}$$

Theorem 1. Let S be a constant operator. Then the condition

$$\|S\| \leqslant \rho, \qquad \rho = e^{c_0\tau}, \tag{12}$$

where c_0 is any constant independent of τ and $|h|$, is necessary and sufficient for stability of the scheme (11) in H.

Necessity. Let the scheme (11) be stable, i.e. (6) is satisfied for all $t = n\tau$, $n = 1, 2, \dots, n_0$. Setting $n = 1$ in (6), we have $\|x_1\| = \|Sx_0\| \leqslant \rho\|x_0\|$, i.e. $\|S\| \leqslant \rho$.

Sufficiency. Let (12) be satisfied. Then $\|x_n\| = \|S^n x_0\| \leqslant \|S\|^n\|x_0\| \leqslant \rho^n\|x_0\| = e^{c_0 t_n}\|x_0\|$, i.e. the estimate (6) is valid. From Theorem 1 and Lemmas 1-6 follow Theorems 2-5 for constant A, B.

Theorem 2. Let $B = B^* > 0$ and $A = A^*$ be independent of t. Then the conditions

$$\frac{1-\rho}{\tau}B \leqslant A \leqslant \frac{1+\rho}{\tau}B, \qquad \rho = e^{c_0\tau} \tag{13}$$

with arbitrary constant c_0 are necessary and sufficient for stability of the scheme (3) in H_B.

Theorem 3. Let $A = A^* > 0$ and $B = B^* > 0$ be independent of t. Then the condition

$$A \leqslant \frac{1+\rho}{\tau}B \quad \text{or} \quad B \geqslant \frac{\tau}{1+\rho}A, \qquad \rho = e^{c_0\tau}, \tag{14}$$

is necessary and sufficient for stability in H_A and H_B with $c_0 \geqslant 0$ ($\rho \geqslant 1$), while conditions (13) are necessary and sufficient for stability of (3) in H_A (and H_B) with $c_0 < 0$ ($\rho < 1$).

Theorem 4. Let $A = A^* > 0$, $B > 0$, and A and B be independent of t. Then the condition

$$A \leqslant \frac{2}{\tau}B \quad \text{or} \quad B \geqslant \frac{1}{2}\tau A \tag{15}$$

necessary and sufficient for the stability of scheme (3) in H_A with constant $c_0 = 0$ ($\rho = 1$).

Theorem 5. Let $B = B^* > 0$ and $A > 0$ not depend on t. Then the condition

$$A^{-1} \geq 0.5\tau B^{-1} \tag{16}$$

is necessary and sufficient for the stability of scheme (3) in H_B with $c_0 = 0$.

Let us note that: 1) Theorems 4 and 5 are proved under the assumption that the operators B and A, respectively, are not self-adjoint; 2) condition (16), unlike conditions (13)–(15), is inconvenient for verification.

6. Conditions (13)–(15) are sufficient for the stability of schemes (3) with operators A and B depending on t, if the operator $A(t) > 0$ (or $B(t) > 0$) satisfies the Lipschitz condition in t with a constant $c_1 > 0$, independent of h and τ:

$$|((A(t) - A(t-\tau))y, y)| \leq \tau c_1 (A(t-\tau)y, y)$$
$$\text{for any } 0 < t = n\tau < t_0,\ y \in H. \tag{17}$$

For constant A and B, scheme (3) reduces to the explicit scheme (5). If $A = A(t)$ and $B = B(t)$, then, introducing $C = C_1(t)$, $x(t+\tau) = A^{1/2}(t)y(t+\tau)$, $\bar{x}(t) = A^{1/2}(t)y(t)$ for $A = A^* > 0$, or $C = C_2(t)$, $x(t+\tau) = B^{1/2}(t)y(t+\tau)$, $\bar{x}(t) = B^{1/2}(t)y(t)$ for $B = B^* > 0$, we transform (3) to the form

$$x(t+\tau) = S(t)\bar{x}(t), \qquad S(t) = E - \tau C(t). \tag{18}$$

Lemma 7. Let $A(t) = A^*(t) > 0$, (or $B(t) = B^*(t) > 0$) and let (17) be satisfied (or the analogous condition for $B(t)$). Then

$$\|\bar{x}(t)\| \leq (1 + 0.5c_1\tau)\|x(t)\| \qquad \text{for any } \tau > 0,\ t > 0,$$

$$\|\bar{x}(t)\| \geq (1 - c_1\tau)\|x(t)\| \qquad \text{for } \tau c_1 < 1,\ t > 0,$$

where $x(t) = A^{1/2}(t-\tau)y(t)$ (or $x(t) = B^{1/2}(t-\tau)y(t)$).

Theorem 6. Let $A(t) = A^*(t) > 0$ and let $A(t)$ satisfy (17), and let $B(t) > 0$ be a non-self-adjoint operator. Then condition (14) with $c_0 \geq 0$ is sufficient for the stability of scheme (3) in $H_{A(t)}$ with constant $\bar{c}_0 = c_0 + 0.5c_1$. If $B(t) = B^*(t) > 0$ and (17) is satisfied for $B(t)$, while $A(t) = A^*(t)$, then condition (13) with arbitrary c_0 is sufficient for the stability of (3) in $H_{B(t)}$.

7. **Theorem 7.** If $A(t) = A^*(t) > 0$, $B(t) = B^*(t) > 0$, then the condition

$$A \geq \frac{1+\rho}{\tau}B \quad \text{for all } t = n\tau \in [0, t_0), \quad \text{where } \rho = e^{c_0\tau\gamma},\ c_0 > 0,\ 0 \leq \gamma < 1, \tag{19}$$

where γ is any nonnegative constant less than one and independent of h and τ, is sufficient for the instability of scheme (3) in $H_{A(t)}$, if $A(t)$ satisfies (17), and for instability in $H_{B(t)}$, if $B(t)$ satisfies condition (17).

8. In (1–3) we used another definition of stability with respect to initial data:

$$\|y(t)\|_{(1)} \leq M_1\|y(0)\|_{(10)} \quad \text{for all } 0 < t = n\tau \leq t_0. \tag{20}$$

It is not difficult to see that (20) follows from (4), since for $c_0 > 0$ one may set $M_1 = e^{c_0 t_0}$, while for $c_0 \leq 0$ we set $M_1 = 1$.

Theorem 8. Let $S = S^*$ be a constant operator. Then condition (12) with $c_0 \geq 0$ is necessary and sufficient for the stability of scheme (5) with $M_1 \geq 1$

in the sense of definition (20). If S is a non-self-adjoint operator, then the condition $\|S\| \leq 1$ is necessary and sufficient for the estimate $\|x(t)\| \leq \|x(0)\|$.

The method presented above makes it possible to investigate the stability of scheme (2) with respect to the right-hand side, as well as stability with respect to perturbations of the operators of the scheme (computational stability of the scheme).

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Note: Figure translations are in progress. See original paper for figures.