ON ACCOUNTING FOR THE EXTENT OF ACTIVE SECTIONS IN THE STUDY OF OPTIMAL TRANSFERS BETWEEN CLOSE, NEAR-CIRCULAR, NON-COPLANAR ORBITS

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Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

Abstract

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ON ACCOUNTING FOR THE EXTENT OF ACTIVE SECTIONS IN THE STUDY OF OPTIMAL TRANSFERS BETWEEN CLOSE, NEAR-CIRCULAR, NON-COPLANAR ORBITS

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In ($^{1-4,7,8}$) the problem of transfers between orbits was considered in an impulsive formulation. In the present work this same problem is considered taking into account the boundedness of the overload (or thrust). The aim of the work is to derive rules for recalculating solutions obtained in the impulsive formulation for the case of extended active sections. The problem is solved under the condition that $n_{\max}$—the maximum overload—is not a small quantity. Under this assumption, the durations of the active sections, for small values of the applied impulses, which according to ($^{1-4,7,8}$) take place, are also small quantities. The entire investigation is carried out with a relative error of the order of the squares of these small quantities.

Fig. 1

Fig. 2

The motion is considered in a cylindrical coordinate system $or\varphi z$, the plane $or\varphi$ of which coincides with the plane of the initial orbit (see Fig. 1). The linearized equations describing the motion in a small neighborhood of a circular orbit of radius r_{cp} are written in the form

$$d\Delta\bar{v}_r/d\bar{t} = 2\Delta\bar{v}_\tau + \Delta\bar{r} + n_r; \quad d\Delta v_\tau/d\bar{t} = -\Delta\bar{v}_r + n_\tau; \quad d\Delta\bar{v}_z/d\bar{t} = -\Delta\bar{z} + n_z, \quad d\bar{q}/d\bar{t} = n;$$

$$d\Delta\bar{r}/d\bar{t} = \Delta\bar{v}_r; \quad d\Delta\bar{z}/d\bar{t} = \Delta\bar{v}_z; \quad d\varphi/d\bar{t} = 1 - \Delta\bar{r} + \Delta\bar{v}_\tau. \tag{1}$$

Here Δv_r, Δv_τ, Δv_z and Δr, Δz denote, respectively, the increments of the radial, transverse, and lateral components of velocity and of the coordinates relative to their values on the circular orbit with $r = r_{\text{cp}}$; φ is the polar angle; t is time; n_r, n_τ, n_z, and n are the components of the reactive acceleration and its modulus, referred to the gravitational acceleration $g(r)$ at $r = r_{\text{cp}}$; $\bar{q} = \bar{c}\ln(m_0/m)$ is the characteristic velocity, where c is the exhaust velocity; m_0 and m are, respectively, the initial and current mass. Bars denote dimensionless quantities. Linear quantities r and z are referred to

to r_{cp}; the velocity components $\dot{r}, v_\tau, v_z, c$ and q are referred to $v_{\text{kp}} = \sqrt{g'(r_{\text{cp}})r_{\text{cp}}}$, the velocity in a circular orbit with $r = r_{\text{cp}}$, and t is referred to $r_{\text{cp}}/v_{\text{kp}} = 1/2\pi$ of the time of a complete revolution in such an orbit. The control functions n_r, n_τ, n_z satisfy the following conditions: a) for limited overload; b) for limited thrust

$$n = \sqrt{n_r^2 + n_\tau^2 + n_z^2} \leq n_{\max}, \qquad P = mg(r_{\text{cp}})n \leq P_{\max}. \tag{2}$$

To solve the problem of minimizing the functional $\Phi = q(t_N)$, the maximum principle of L. S. Pontryagin (^{5},^{6}) is used. The function H is written in the form

$$\begin{aligned} H = [(\mathbf{sn}) + p_q n] - (s_\tau - p_r)\Delta\bar{v}_r + \\ + (2s_r + p_\varphi)\Delta\bar{v}_\tau + p_z\Delta\bar{v}_z + (s_r - p_\varphi)\Delta\bar{r} - s_z\Delta\bar{z} + p_\varphi, \end{aligned} \tag{3}$$

where $\mathbf{n} = (n_r, n_\tau, n_z)$, $\mathbf{s} = (s_r, s_\tau, s_z)$ are vectors, and $s_r, s_\tau, s_z, p_q, p_r, p_z$ and p_φ denote the conjugate variables corresponding to $\Delta v_r, \Delta v_\tau, \Delta v_z, q, \Delta r, \Delta z$ and φ. The conditions for a minimum of Φ have the form

$$n_r = ns_r/s; \quad n_\tau = ns_\tau/s, \quad n_z = ns_z/s; \quad n = \begin{cases} n_{\max} & \text{for } \vartheta > 0, \\ 0 & \text{for } \vartheta < 0, \end{cases} \tag{4}$$

where $s = \sqrt{s_r^2 + s_\tau^2 + s_z^2}$; $\vartheta = s + p_q$ is the switching function;

$$\frac{ds_r}{d\bar{t}} = s_\tau - p_r; \qquad \frac{ds_\tau}{d\bar{t}} = -2s_r - p_\varphi; \qquad \frac{dp_r}{d\bar{t}} = -s_r + p_\varphi;$$

$$\frac{dp_q}{d\bar{t}} = 0, \qquad \frac{ds_z}{d\bar{t}} = -p_z; \qquad \frac{dp_z}{d\bar{t}} = s_z; \qquad \frac{dp_\varphi}{d\bar{t}} = 0. \tag{5}$$

For the minimum of $q(\bar{t}_N)$, the function p_q at $\bar{t} = \bar{t}_N$ must be equal to -1. The general solution of system (5) is written in the form

$$s_r = A\cos t + B\sin t - 2p_\varphi; \qquad s_\tau = -2A\sin t + 2B\cos t + 3p_\varphi t + C,$$

$$s_z = D\cos t + E\sin t; \qquad p_q = -1; \qquad p_\varphi = \text{const}, \tag{6}$$

where A, B, C, D, E and p_φ are arbitrary constants. In the case of limited thrust all relations (3)–(6) remain valid, except for the equation for p_q, which is written in the form

$$\frac{dp_q}{d\bar{t}} = -\frac{e^{\bar{q}/\bar{c}}}{\bar{c}m_0 g(r_{\text{cp}})}P\vartheta, \qquad \text{where } P = \begin{cases} P_{\max} & \text{for } \vartheta > 0, \\ 0 & \text{for } \vartheta < 0, \end{cases} \qquad p_q(\bar{t}_N) = -1. \tag{7}$$

A singular control, for which $\vartheta \equiv 0$ and the magnitude of overload or thrust can be regulated, by virtue of (6), can occur then and only then when either

$$p_\varphi = C = 0; \qquad s_z = \pm\sqrt{3}\, s_r; \qquad A^2 + B^2 = 1/4, \tag{8}$$

or

$$p_\varphi = A = B = D = E = 0; \qquad C = 1. \tag{9}$$

It is seen from (6) that if a singular control is possible, then it exists over the entire time interval under consideration. In the case of the problem of transfers between orbits with an unspecified angular range, considered in ([3],[4],[7],[8]), $p_\varphi = 0$, and the singular controls correspond to the so-called degenerate transfers, when the necessary optimality conditions for the instants of impulse application do not determine them.

The functions $\Delta\bar{r}(\bar{t})$ and $\Delta\bar{z}(\bar{t})$, which determine the deviation from the initial orbit, by means of equations (1) and (4) can be written in the form

$$\Delta\bar{r}(\bar{t}) = \int_{\bar{t}_0}^{\bar{t}} \{s_r(\xi)\sin(\bar{t} - \xi) + 2s_\tau(\xi)[1 - \cos(\bar{t} - \xi)]\}\, \frac{n(\xi)}{s(\xi)}\, d\xi,$$

$$\Delta\bar{z}(\bar{t}) = \int_{\bar{t}_0}^{\bar{t}} s_z(\xi)\sin(\bar{t} - \xi)\,\frac{n(\xi)}{s(\xi)}\, d\xi, \tag{10}$$

where ξ is the variable of integration, and $n(\xi) = 0$ everywhere outside the interval $\bar{t}_0 \le \bar{t} \le \bar{t}_N$. On the other hand, if the terminal orbit is specified, then for $\bar{t} \ge \bar{t}_N$ the functions $\Delta\bar{r}(\bar{t})$ and $\Delta\bar{z}(\bar{t})$ are determined by the formulas

$$\Delta\bar{r}(\bar{t}) = \Delta_0 + \Delta_c \cos\bar{t} + \Delta_s \sin\bar{t}; \qquad \Delta\bar{z}(\bar{t}) = \Delta_z \sin\bar{t}, \tag{11}$$

where $\Delta_0, \Delta_c, \Delta_s$ and Δ_z are known constants ([3],[7],[8]). In the problem of transfers

between the orbits, the functions (10) and (11) for $\bar{t} \ge \bar{t}_N$ must coincide. This gives a system of boundary conditions.

$$\int_{\bar{t}_0}^{\bar{t}_N} (s_r \sin\xi + 2s_\tau \cos\xi)\, \frac{n}{s}\, d\xi = -\Delta_c; \qquad \int_{\bar{t}_0}^{\bar{t}_N} (s_r \cos\xi - 2s_\tau \sin\xi)\, \frac{n}{s}\, d\xi = \Delta_s,$$

$$2\int_{\bar{t}_0}^{\bar{t}_N} \frac{s_\tau n}{s}\, d\xi = \Delta_0; \qquad \int_{\bar{t}_0}^{\bar{t}_N} \frac{s_z n}{s} \sin\xi\, d\xi = 0; \qquad \int_{\bar{t}_0}^{\bar{t}_N} \frac{s_z n}{s} \cos\xi\, d\xi = \Delta_z. \tag{12}$$

These 5 conditions serve to determine the 5 unknown constants A, B, C, D, E, $p_\varphi = 0$, which enter the expressions for the adjoint variables. In what follows we shall consider the problem in which the instants $\bar{t}_0$ and $\bar{t}_N$ are determined optimally. It is shown that for this they must be, respectively, the beginning and the end of the active arcs determined according to (4).

The investigation of transfers between orbits for which special controls are possible, carried out on the basis of conditions (8) and (9), showed that for any thrust-control laws ensuring satisfaction of the boundary conditions (12), the required value of the characteristic velocity is one and the same and coincides with its value obtained in ([3],[4],[7],[8]) as a result of solving the problem in the impulsive formulation. However, the largest domain of the parameters $\Delta_0, \Delta_c, \Delta_s$ and Δ_z for which transfers with special controls are possible depends on the thrust-control law and is greatest for the boundary control law containing no more than two active arcs within one revolution. The boundaries of the active arcs in this case are determined from equality (12).

Let $\bar{t}_k^-, \bar{t}_k, \bar{t}_k^+$ denote the beginning, middle, and end of the k-th active arc. $\bar{t}_0^- = \bar{t}_0$, $\bar{t}_N^+ = \bar{t}_N$. Conditions (12) can be rewritten in the form:

$$n_{\max} \sum_{k=0}^{N} \int_{\bar{t}_k^-}^{\bar{t}_k^+} \frac{s_r \sin\xi + 2s_\tau \cos\xi}{s}\, d\xi = -\Delta_c;$$

$$n_{\max}\sum_{k=0}^{N}\int_{\bar t_k^-}^{\bar t_k^+}\frac{s_r\cos\xi-2s_\tau\sin\xi}{s}\,d\xi=\Delta_s,\qquad n_{\max}\sum_{k=0}^{N}\int_{\bar t_k^-}^{\bar t_k^+}\frac{s_\tau}{s}\,d\xi=\frac{\Delta_0}{2};\tag{13}$$

$$n_{\max}\sum_{k=0}^{N}\int_{\bar t_k^-}^{\bar t_k^+}\frac{s_z\sin\xi}{s}\,d\xi=0;\qquad n_{\max}\sum_{k=0}^{N}\int_{\bar t_k^-}^{\bar t_k^+}\frac{s_z\cos\xi}{s}\,d\xi=\Delta_z.$$

We shall further use the smallness of the lengths $\Delta\bar t_k=\bar t_k^+-\bar t_k^-$ of the active arcs. Let $f(\xi)$ denote any one of the integrand expressions in the equalities (13). Then, expanding $f(\xi)$ in a series in a neighborhood of $\xi=\bar t_k$, we obtain

$$\int_{\bar t_k^-}^{\bar t_k^+}f(\xi)\,d\xi=f(\bar t_k)\Delta\bar t_k+O(\Delta\bar t_k^3).\tag{14}$$

In accordance with (4) and (6), the ends of the active arcs are determined from the equalities $s(t_k^+)=s(t_k^-)=1$. Expanding the left-hand sides of these equalities in a series in a neighborhood of $\bar t=\bar t_k$ and denoting here and below the values of all functions at this instant by the subscript k below, we shall have

$$s_k+\frac{s_k'}{2}\Delta\bar t_k+\frac{s_k''}{8}\Delta\bar t_k^2+O(\Delta\bar t_k^2)=1;\qquad s_k-\frac{s_k'}{2}\Delta\bar t_k+\frac{s_k''}{8}\Delta\bar t_k^2+O(\Delta\bar t_k^3)=1,$$

$$s'(\bar t_k)=0+O(\Delta\bar t_k^2);\qquad s(\bar t_k)=1+O(\Delta\bar t_k^2).\tag{15}$$

Equalities (13), taking (14) and (15) into account, can be rewritten in the form

$$\sum_{k=0}^{N}\left[\left(s_{r_k}\sin\bar t_k+2s_{\tau_k}\cos\bar t_k\right)+O\left(\frac{\Delta\bar v_k^2}{n_{\max}^2}\right)\right]\Delta\bar v_k=-\Delta_c;$$

$$\sum_{k=0}^{N}\left[\left(s_{r_k}\cos\bar t_k-2s_{\tau_k}\sin\bar t_k\right)+O\left(\frac{\Delta\bar v_k^2}{n_{\max}^2}\right)\right]\Delta\bar v_k=\Delta_s,\tag{16}$$

$$\sum_{k=0}^{N}\left[s_{\tau_k}+O\left(\frac{\Delta\bar v_k^2}{n_{\max}^2}\right)\right]\Delta\bar v_k=\frac{\Delta_0}{2};\qquad \sum_{k=0}^{N}\left[s_{z_k}\sin\bar t_k+O\left(\frac{\Delta\bar v_k^2}{n_{\max}^2}\right)\right]\Delta\bar v_k=0;$$

$$\sum_{k=0}^{N}\left[s_{z_k}\cos\bar t_k+O\left(\frac{\Delta\bar v_k^2}{n_{\max}^2}\right)\right]\Delta\bar v_k=\Delta_z,$$

where

$$\Delta\bar{v}_k = n_{\max}\Delta\bar{t}_k; \quad s_{r_k} = A\cos\bar{t}_k + B\sin\bar{t}_k; \quad s_{q_k} = -2A\sin\bar{t}_k + 2B\cos\bar{t}_k + C;$$

$$s_{z_k} = D\cos\bar{t}_k + E\sin\bar{t}_k.$$

Equalities (15) and (16) serve to determine the instants $\bar{t}_k$ and the constants A, B, C, D, and E. With a relative error of order $\Delta\bar{t}_k^2$, these equalities coincide with the equalities obtained in solving the problem in the impulsive formulation. It can be proved that this error does not increase if the direction of thrust for $\bar{t}_k^- \leq t \leq \bar{t}_k^+$ is taken to coincide with the direction of thrust at $\bar{t} = \bar{t}_k$ $(k = 0, 1, \ldots, N)$, which coincides with the direction of the k-th impulse.

What has been set forth above can be formulated in the form of the following rules for recalculating impulsive solutions for the case of limited overload:

1) The midpoints of the active arcs coincide with the instants at which the impulses are applied.
2) The lengths of the active arcs are determined by the formula $\Delta\bar{t}_k = \Delta\bar{v}_k/n_{\max}$, where $\Delta\bar{v}_k$ is the magnitude of the impulse.
3) The constants A, B, C, D, and E coincide with the Lagrange multipliers determined in solving the problem in the impulsive formulation.
4) The direction of thrust on each of the active arcs coincides with the direction of the corresponding impulse.

The case of limited thrust differs from the case of limited overload by the equation for p_q. However, since $\max\vartheta(\bar{t}) = O(\Delta\bar{t}_k^2)$, see Fig. 2, it follows, by virtue of (7), that $p_q = -1 + O(\Delta\bar{t}_k^2)$. Consequently, all the results formulated above are preserved, except for the formula for $\Delta\bar{t}_k$. In the case of limited thrust the expression for $\Delta\bar{t}_k$ has the form

$$\Delta\bar{t}_k = \frac{m_{0g}(r_{\text{cp}})}{P_{\max}} \exp\left[-\frac{1}{c}\sum_{i=0}^{k-1}\Delta\bar{v}_i\right] \frac{\Delta\bar{v}_k}{1 + \Delta\bar{v}_k/2c} \qquad (k = 0, 1, \ldots, N).$$

The relative error of the parameters of the optimal transfers determined in accordance with these rules is, in the general case, a quantity of order $\Delta\bar{v}_k^2/n_{\max}^2$ and, consequently, for not small values of $n_{\max}$ lies outside the accuracy of the linearized equations of motion.

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Note: Figure translations are in progress. See original paper for figures.