BOUNDARY-VALUE PROBLEMS OF S. L. SOBOLEV TYPE FOR ELLIPTIC OPERATORS

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 517.946.9

MATHEMATICS

B. STERNIN

BOUNDARY-VALUE PROBLEMS OF S. L. SOBOLEV TYPE FOR ELLIPTIC OPERATORS

(Presented by Academician I. G. Petrovskii on 29 III 1968)

1. Let X be a manifold (closed, for simplicity) and let $Y \subset X$ be its submanifold of codimension $\nu \geq 1$.

By problems of Sobolev type, or, briefly, S-problems, we shall mean the problem of finding a function satisfying an elliptic equation on $X \setminus Y$ and assuming on Y certain boundary values.

As far as I know, problems of this type were first considered by S. L. Sobolev (3), who proved unique solvability of the Dirichlet problem for the polyharmonic equation $\Delta^m u = 0$. In the paper (4) a general theory of such problems was constructed for the case of closed submanifolds of an arbitrary (elliptic) pseudodifferential operator on the manifold X and arbitrary pseudodifferential operators on the submanifold Y. More precisely, in that paper spaces and conditions on the coefficients of the equation were indicated that are necessary and sufficient in order that the S-problem in these spaces be normally solvable, i.e., uniquely solvable up to finite-dimensional spaces.

In the present paper an analogous question is investigated for the case when the submanifold Y of the S-problem has a boundary.

If we realize pseudodifferential expressions as an unbounded operator acting in spaces in which the corresponding S-problem is normally solvable, then such an operator will, obviously, be Fredholm, i.e., will have finite-dimensional kernel and cokernel. Thus, the theorem on normal solvability of such a problem may be regarded as a finiteness theorem for the corresponding operator. The topological aspects of boundary S-problems, and in particular the computation of the index of the corresponding (Fredholm) operator, I shall consider elsewhere.

2. Thus, let X be (for simplicity) a closed manifold of dimension N, and let $Y \subset X$ be its submanifold of codimension ν:

$$1 \leq \nu \leq \dim X - 1.$$

The cases $\nu = \dim X$, $\nu = \dim X - 1$ are atypical and differ somewhat from the general case. For example, if $\nu = \dim X$, i.e., the manifold Y degenerates to a point, then no boundary values can be prescribed on it, and nevertheless the problem will be normally solvable. If $\nu = \dim X - 1$, then the boundary of the manifold Y degenerates to a point, and nothing can be prescribed on it. Next, we note that for $\nu > 1$, normally solvable problems are only S-problems (at least in Sobolev spaces), whereas in the case $\nu = 1$, besides problems of type S, one may also consider other problems (of conjugation-problem type).

3. In the paper (^{5}) we introduced the notions of boundary and coboundary operators. Let us recall them.

For any finite-dimensional complex vector bundle $E \to X$ over a smooth (possibly with boundary) manifold X and any real number $s \in R^1$, we shall denote by $\Gamma^s(X, E)$ the Sobolev space (^{2}) of sections of the bundle E.

If $i : Y \subset X$ is the embedding of a submanifold Y of codimension ν in a manifold X, then for $s > \nu/2$ there is a continuous mapping

$$\delta = \delta_Y : \Gamma^s(X, E) \to \Gamma^{s-\nu/2}(Y, i^*E),$$

which assigns to each section $f \in \Gamma^s(X, E)$ its restriction $f|Y$ to the manifold Y. We shall call such an operator an **elementary boundary operator**. A **general boundary operator** is a composition

$$\Gamma^s(X, E_1) \xrightarrow{D} \Gamma^t(X, E_2) \xrightarrow{\delta} \Gamma^{t-\nu/2}(Y, i^*E_2)$$

of some pseudodifferential operator D and the restriction operator. It is easy to see that a boundary operator is continuous if its components are continuous.

The operator adjoint (in the sense of the L_2-scalar product) to an elementary boundary operator is

$$\delta^* = \delta^*_Y : \Gamma^{-s+\nu/2}(Y, i^*E) \to \Gamma^{-s}(X, E),$$

which we shall call an **elementary coboundary operator**. A **general coboundary operator** is a composition

$$\Gamma^{-s+\nu/2}(Y, i^*E_1) \xrightarrow{\delta^*} \Gamma^{-s}(X, E_1) \xrightarrow{D} \Gamma^{-t}(X, E_2)$$

of an elementary coboundary operator and some pseudodifferential operator D.

Let us now consider the problem

$$D_X u = f_X, \qquad x \notin Y \cup \partial Y \tag{1}$$

of finding a section $u \in \Gamma^s(X, E_1)$ satisfying, off the manifold Y, equation (1) for $f_X \in \Gamma^{s-m}(X, E_2)$, where m is the order of the expression D_X. First of all, note that if $s - m + \nu/2 \geq 0$, then equation (1) is equivalent to the equation on the whole manifold X, and, thus, problem (1) is normally solvable if and only if the expression D_X is elliptic.

Now let $s - m + \nu/2 < 0$. Then, since in the space $\Gamma^{s-m}(X, E_2)$ there are nontrivial elements concentrated on the submanifold Y, and since $\dim Y > 0$, equation (1) has, generally speaking, an infinite-dimensional space of solutions. To remove this nonuniqueness one must additionally impose a certain number of boundary conditions. In order to understand the situation precisely, we proceed (following the principle of locality) as follows. We shall straighten the boundary near a typical point of the submanifold Y. Then we arrive at a problem in the space $\mathbf{R}^N$ and see that on the manifold Y one must impose a certain number of boundary operators connected with the operator D_X by a certain algebraic condition (see $(^4)$). In the case when the manifold Y had no edge, this condition, together with the ellipticity condition of the operator D_X, ensured normal solvability of the equation. In invariant terms, fulfillment of this condition means that normal solvability of our problem is equivalent to normal solvability of two problems: the problem for an elliptic operator on the manifold X and the problem for an elliptic operator on the manifold Y. In the case when the manifold Y has no boundary (recall that we have everywhere assumed that the manifold X is closed), these problems are normally solvable. If, however, the manifold Y has an edge ∂Y and $\dim \partial Y > 0$, this is, generally speaking, no longer the case. To understand the situation, we again consider a typical point on the (closed) manifold ∂Y. Again, in accordance with the principle of locality, we straighten the boundary near such a point. Let $t^1, \dots, t^\nu, x^1, \dots, x^{n-1}, x^n = (t, x', x_n)$ be coordinates in the space $\mathbf{R}^n$, with the (open) manifold Y is given by the equation $t = 0,\ x^n > 0$. Freezing the coefficients of equation (1) and passing in it to the Fourier image with respect to the variables $x^1, \dots, x^{n-1} = x'$, we obtain an equation which we shall conventionally write in the form

$$D_X(t, \xi', x_n) u(t, \xi', x_n) = f. \tag{2}$$

This equality holds for all x^n and t, except for points lying on the half-line $t = 0,\ x^n > 0$. For the subsequent investigation the right-hand side of equation (2) is inessential, and we shall assume $f(t, \xi', x_n) \equiv 0$. Hence it follows that in the whole space $\mathbb{R}^N$ equation (2) can be written in the form

$$D_X(t, \xi', x^n) u(t, \xi', x_n) = C v(\xi', x_n), \tag{3}$$

where the matrix C has the form

$$C = \begin{pmatrix} 1 & & \frac{\partial^j \delta(t)}{\partial t^j} & \\ & \ddots & & \ddots \\ & & \ddots & \\ & & \frac{\partial^j \delta(t)}{\partial t^j} & \end{pmatrix}.$$

Here δ is the Dirac δ-function; $j = (j_1, \dots, j_\nu)$, $|j| = \sum j_k \le l^\nu$ (see (5)), $v(\xi', t)$ is a (vector-)function concentrated on the half-axis $x^n > 0$ with as yet arbitrary components. Let now $\xi' \neq 0$. Then the solution of equation (3) has the form

$$u(t, \xi', x^n) = \int_{\mathbb{R}^1} \int_{\mathbb{R}^{n-1}} \exp\{-i[(t,\tau)+(x^n,\xi_n)]\} D_X^{-1}(\tau, \xi', \xi_n) C(\tau) v(\xi', \xi_n)\, d\tau\, d\xi_n.$$

Applying to the left- and right-hand sides of the written equality the boundary operators $\delta_Y D_Y(t, \xi', x^n)$, we obtain, for any $\xi' \neq 0$,

$$g = \frac{1}{2\pi} \int_{\mathbb{R}^1} \exp[-i(x^n, \xi_n)] \int_{\mathbb{R}^{n-1}} D_Y(\tau, \xi', \xi_n) D_X^{-1}(\tau, \xi', \xi_n) C(\tau)\, d\tau\, v(\xi', \xi_n)\, d\xi_n,$$

where $g = \delta_Y D_Y u$. Thus we arrive at a system of pseudodifferential equations

$$g(x^n) = \frac{1}{2\pi} \int_{\mathbb{R}^1} \exp[-i(x^n, \xi_n)] \sigma(\xi', \xi_n) v(\xi', \xi_n)\, d\xi_n \tag{4}$$

with symbol

$$\sigma(\xi', \xi_n) = \int_{\mathbb{R}^{n-1}} D_Y(\tau, \xi', \xi_n) D_X^{-1}(\tau, \xi', \xi_n) C(\tau)\, d\tau.$$

The unique solvability of equation (4) is what interests us.

Let us first observe that, since the solutions $v(\xi', \xi_n)$ of this equation must be concentrated for $x^n \ge 0$, the equation obtained is essentially an equation on a half-line.

From what was said above it follows that the symbol of this operator is not equal to zero for any ξ_n. (Recall that $\xi' \neq 0$.) The theory of such equations in Sobolev spaces was developed in a series of works by M. I. Vishik and G. I. Eskin (^{1}). This theory is based on the principle of factorization and consists in the following. It is known that the matrix $\sigma(\xi', \xi_n)$ admits the factorization

$$\sigma(\xi', \xi_n) = \sigma_+(\xi', \xi_n) \sigma_-(\xi', \xi_n)$$

with factors nondegenerate for $\xi' \neq 0$ and analytic, respectively, in the upper and lower complex half-planes. It is shown—

It follows from (1) that the operator Σ_+ with symbol $\sigma_+(\xi', \xi_n)$ is Fredholm. Let $\dim \operatorname{Ker} \Sigma_+$ and $\dim \operatorname{Coker} \Sigma_+$ be the dimensions of its kernel and cokernel. Then, specifying $\dim \operatorname{Ker} \Sigma_+$ boundary and $\dim \operatorname{Coker} \Sigma_+$ coboundary conditions at the point $x_n = 0$, connected by a natural algebraic condition of the Shapiro–Lopatinskii type, we obtain unique solvability of equation (4).

4. In the preceding paragraph we have completely defined the boundary-value problem that we shall call a boundary-value problem of type S, or, briefly, an S-problem. (In order not to burden the exposition, I do not write out the spaces of the right-hand sides of the S-problem. The reader will have no difficulty filling this gap.)

We now formulate the main result of the paper.

Theorem. *Suppose that the algebraic conditions on the coefficients of the problem formulated in § 3 are satisfied. Then the boundary S-problem is normally solvable.*

5. Lack of space does not allow us to consider constructions that generalize S-problems and are of great importance in considering topological aspects of the theory, namely elliptic morphisms ([7]) for manifolds with boundary. These questions will be considered in a subsequent paper.

I discussed the questions touched upon in this work with V. V. Grushin; I express my gratitude to him for his attention.

Institute for Problems in Mechanics
Academy of Sciences of the USSR

Received
13 XII 1967

REFERENCES

1. M. I. Vishik, G. I. Eskin, *Matem. sborn.*, **74**, issue 3 (1967).
2. R. Palais, *Seminar on the Atiyah—Singer Index Theorem*, Princeton, 1963–1964, Princeton, 1965.
3. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.
4. B. Yu. Sternin, *Tr. Mosk. matem. obshch.*, **15**, 346 (1966).
5. B. Yu. Sternin, *DAN*, **172**, No. 1 (1967).

Note: Figure translations are in progress. See original paper for figures.