ON AN INVARIANT TRANSFORMATION OF THE EQUATIONS OF PLANE STEADY MOTIONS OF AN IDEAL GAS

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ON AN INVARIANT TRANSFORMATION OF THE EQUATIONS OF PLANE STEADY MOTIONS OF AN IDEAL GAS

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A transformation of the variables and unknown functions is constructed for the equations of plane steady motions of a gas, leaving the form of these equations unchanged. Under application of this transformation, the initial equation of state of the gas is transformed into a new one containing arbitrary parameters. This may make it possible, by choosing the parameters, to approximate complicated equations of state of a given medium (in particular, for gases taking account of equilibrium reactions occurring in them) and, using the indicated transformation, to reduce the problem to one for a gas with a simpler equation of state, for which it can be solved by less complicated means. In contrast to existing works in this direction, the present approach makes it possible to consider vortical motions of a gas, as well as flows with shock waves.

The equations of plane stationary motions of a gas ([1])

$$\rho_1 u_1 \frac{\partial u_1}{\partial x_1} + \rho_1 v_1 \frac{\partial u_1}{\partial y_1} + \frac{\partial p_1}{\partial x_1} = 0, \qquad \rho_1 u_1 \frac{\partial v_1}{\partial x_1} + \rho_1 v_1 \frac{\partial v_1}{\partial y_1} + \frac{\partial p_1}{\partial y_1} = 0,$$

$$\frac{\partial}{\partial x_1}(\rho_1 u_1) + \frac{\partial}{\partial y_1}(\rho_1 v_1) = 0 \tag{1}$$

allow one to introduce into consideration new independent functions $x_2(x_1, y_1)$, $y_2(x_1, y_1)$, for which the differential relations

$$dx_2 = [1 + a(\rho_1 v_1^2 + p_1)]\, dx_1 - a\rho_1 u_1 v_1\, dy_1,$$

$$dy_2 = -a\rho_1 u_1 v_1\, dx_1 + [1 + a(\rho_1 u_1^2 + p_1)]\, dy_1. \tag{2}$$

hold. Here u_1, v_1 are the projections of the gas velocity on the coordinate axes x_1, y_1; p_1, ρ_1 are the pressure and density of the gas; a is an arbitrary constant.

In system (1), passing to the new independent variables x_2, y_2 by formulas (2), one can obtain an analogous system

$$\rho_2 u_2 \frac{\partial u_2}{\partial x_2} + \rho_2 v_2 \frac{\partial u_2}{\partial y_2} + \frac{\partial p_2}{\partial x_2} = 0, \qquad \rho_2 u_2 \frac{\partial v_2}{\partial x_2} + \rho_2 v_2 \frac{\partial v_2}{\partial y_2} + \frac{\partial p_2}{\partial y_2} = 0,$$

$$\frac{\partial}{\partial x_2}(\rho_2 u_2) + \frac{\partial}{\partial y_2}(\rho_2 v_2) = 0. \tag{3}$$

Here

$$u_2 = u_1/\Delta_1, \qquad v_2 = v_1/\Delta_1, \qquad p_2 = p_1/\Delta_1, \qquad \rho_2 = \rho_1/\delta_1,$$

$$\Delta_1 = 1 + ap_1, \qquad \delta_1 = 1 + a(\rho_1 u_1^2 + \rho_1 v_1^2 + p_1). \tag{4}$$

Thus, the indicated transformations (2), (4) put into correspondence with the flow plane $x_1 y_1$ (plane 1) a new flow plane $x_2 y_2$ (plane 2).

It is easy to see that the inverse transition from plane 2 to plane 1 can be effected by the formulas

$$\begin{gathered} dx_1 = \left[1 - a(\rho_2 v_2^2 + p_2)\right] dx_2 + a\rho_2 u_2 v_2 dy_2, \\ dy_1 = a\rho_2 u_2 v_2 dx_2 + \left[1 - a(\rho_2 u_2^2 + p_2)\right] dy_2, \\ u_1 = u_2/\Delta_2, \qquad v_1 = v_2/\Delta_2, \qquad p_1 = p_2/\Delta_2, \qquad \rho_1 = \rho_2/\delta_2, \\ \Delta_2 = 1 - ap_2, \qquad \delta_2 = 1 - a(\rho_2 u_2^2 + \rho_2 v_2^2 + p_2). \end{gathered} \tag{5}$$

The transformations (2), (4) carry the streamlines $\psi_1(x_1, y_1) = \text{const}$ of plane 1 into the streamlines $\psi_2(x_2, y_2) = \text{const}$ of plane 2. Indeed, since

$$\frac{v_1}{u_1} = \frac{v_2}{u_2}, \qquad \frac{\partial \psi_1}{\partial x_1} \Big/ \frac{\partial \psi_1}{\partial y_1} = \frac{\partial \psi_2}{\partial x_2} \Big/ \frac{\partial \psi_2}{\partial y_2},$$

then along the streamlines $d\psi_1 = \Delta_1 d\psi_2$. The asserted statement follows from this. The same can also be said about the lines orthogonal to the streamlines of both planes. The mapping obtained is not conformal; there is only conservatism of the angles between the streamlines and the trajectories orthogonal to them.

The vortices of the two planes are related by the relation

$$\operatorname{rot}_1 \mathbf{V}_1 = \Delta_1 \delta_1 \operatorname{rot}_2 \mathbf{V}_2 \qquad (\operatorname{rot}_i \mathbf{V}_i = \partial v_i/\partial x_i - \partial u_i/\partial y_i, \quad i = 1, 2), \tag{6}$$

which can be obtained from the system (1) and formulas (2), (4).

It is clear from relation (6) that in both planes simultaneously the motion is either vortical or potential.

Let us dwell somewhat more closely on the equations of state of the two planes. If in plane 1 the relation $F_1(p_1, \rho_1, \psi_1) = 0$ is fixed, then in plane 2 the corresponding relation $F_2(p_2, \rho_2, \psi_2) = 0$ will be completely determined. Indeed, from the formula $\rho_2 = \rho_1/\delta_1$ we have

$$u_1^2 + v_1^2 = \frac{\Delta_1}{a}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right). \tag{7}$$

Differentiating (7) along a streamline and using the Bernoulli equation in differential form, for plane 1 one may write the relation

$$\frac{d\rho_1}{\rho_1^2} + \frac{a}{1+ap_1}\frac{dp_1}{\rho_1} = \frac{d\rho_2}{\rho_2^2} - \frac{a}{1-ap_2}\frac{\partial p_2}{\rho_2}. \tag{8}$$

Expressing ρ_1 in terms of p_1, ψ_1 from $F_1(p_1, \rho_1, \psi_1) = 0$ and using formula (4) for p_2, from the differential equation (8) one can obtain ρ_2 as a function of p_1, ψ_1. Then, upon carrying out (5) for p_1 and the equality $\psi_1(x_1, y_1) = \psi_2(x_2, y_2) + \text{const}$, the desired relation $F_2(p_2, \rho_2, \psi_2) = 0$ will be determined.

Assuming that $dp_1/d\rho_1 > 0,\ dp_2/d\rho_2 > 0$, introduce the notation

$$dp_1/d\rho_1 = c_1^2, \qquad dp_2/d\rho_2 = c_2^2, \qquad (u_1^2+v_1^2)/c_1^2 = \mathrm{M}_1^2, \qquad (u_2^2+v_2^2)/c_2^2 = \mathrm{M}_2^2. \tag{9}$$

Here c_1, c_2 are the speeds of sound in the two planes, and M_1, M_2 are the Mach numbers.

When the equality $dp_1/dp_2 = \Delta_1^2$, formulas (4) for u_1, v_1 and relation (7) are used, from equation (8) the equality can be obtained without particular difficulty:

$$1 - \mathrm{M}_2^2 = \lambda^2(1 - \mathrm{M}_1^2) \qquad (\lambda = \rho_2/\rho_1). \tag{10}$$

Hence it follows that in the corresponding domains of the two planes the gas motion is either subsonic, or sonic, or supersonic.

If in plane 1 the solution of some problem is known for a definite-

state equation, the solution of the corresponding problem in plane 2 for another equation of state can be obtained by a simple recalculation. Indeed, if the known functions $u_1(x_1, y_1)$, $v_1(x_1, y_1)$, $p_1(x_1, y_1)$, $\rho_1(x_1, y_1)$ are available, the transformation formulas (2) give

$$
\begin{aligned}
x_2 &= \int_L [1 + a(\rho_1 v_1^2 + p_1)]\, dx_1 - a\rho_1 u_1 v_1\, dy_1 + \text{const}, \\
y_2 &= \int_L (-a\rho_1 u_1 v_1)\, dx_1 + [1 + a(\rho_1 u_1^2 + p_1)]\, dy_1 + \text{const}.
\end{aligned} \tag{11}
$$

Here the contour of integration plays no role, since the expressions under the integrals are complete differentials.

From relations (4), determining the functions $u_2(x_1, y_1)$, $v_2(x_1, y_1)$, $p_2(x_1, y_1)$, $\rho_2(x_1, y_1)$ for $x_2 = x_2(x_1, y_1)$, $y_2 = y_2(x_1, y_1)$, we obtain the solution of the corresponding problem in plane 2 in parametric form.

If in plane 1 the shape of the shock wave $f(x_1, y_1) = 0$ is specified, then the corresponding image of the shock wave in plane 2 can be found from (11) by eliminating x_1, y_1 from the equations $x_2 = x_2(x_1, y_1)$, $y_2 = y_2(x_1, y_1)$, $f(x_1, y_1) = 0$.

Generalization of S. A. Chaplygin' s approximation to the case of vortical gas motion. First of all it should be noted that if, in the equations of plane steady gas motions,

$$
\rho_1 \frac{du}{dt} + \frac{\partial p}{\partial x} = 0, \qquad \rho \frac{dv}{dt} + \frac{\partial p}{\partial y} = 0, \qquad \operatorname{div}(\rho \mathbf{V}) = 0, \qquad \left[\frac{d}{dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right] \tag{12}
$$

one makes the replacement of the flow parameters

$$
\rho = \chi^2(\psi)\rho', \qquad u = u'/\chi(\psi), \qquad v = v'/\chi(\psi), \qquad p = p', \tag{13}
$$

then system (12) will be invariant with respect to the transformations (13), i.e.

$$
\rho' \frac{du'}{dt} + \frac{\partial p'}{\partial x} = 0, \qquad \rho' \frac{dv'}{dt} + \frac{\partial p'}{\partial y} = 0, \qquad \operatorname{div}(\rho' \mathbf{V}') = 0 \qquad \left(\frac{d\chi(\psi)}{dt} = 0\right).
$$

Here $\chi(\psi)$ is an arbitrary function of the flow.

Consider in plane 1 the vortical motion of an incompressible fluid ($\rho_1 = \text{const}$). Passing from plane 1 to plane $1'$ by means of transformations (13), we have $\rho_1' = \rho_1/\chi^2(\psi)$.

Suppose that transformations (2), (4) take plane $1'$ into plane 2. (Since under this mapping the streamlines pass into themselves, the index of the function ψ has been omitted.) Along streamlines ρ_1' is constant, therefore $M_1' = 0$ in plane $1'$. According to equality (10), $M_2 = \sqrt{1 - \lambda^2} < 1$ ($\lambda < 1$), i.e. in plane 2 the gas motion must be subsonic. Applying Bernoulli' s equation to equation (7) for plane $1'$, $u_1'^2 + v_1'^2 + 2p_1'/\rho_1' = 2i_1'$ (i_1' is a function of the flow, and $i_1'\rho_1' = i_1\rho_1$) and formula (5) ($p_1 = p_2/\Delta_2$), we obtain the function

$$P_2 = A' - B'/\rho_2 \qquad \left(A' = \frac{1 + 2ai_1'\rho_1'}{2a(1 + ai_1'\rho_1')}, \quad B' = \frac{\rho_1'}{2a(1 + ai_1'\rho_1')}\right),$$

which in the parameters of plane 1 has the form

$$P_2 = A - B/\rho_2 \qquad \left(A = \frac{1 + 2ai_1\rho_1}{2a(1 + ai_1\rho_1)}, \quad B = \frac{\rho_1}{2a(1 + ai_1\rho_1)\chi^2}\right). \tag{14}$$

Thus, transformations (2), (4) put the motion of an incompressible fluid (plane 1) into correspondence with the motion of a compressible fluid with equation of state (14) (plane 2). The expressions A and B contain two functions of the flow, $\chi(\psi)$, $i_1(\psi)$, which can be chosen from the conditions of appro-

approximation of the Poisson adiabat $p_2 = \theta(\psi)\rho_2^\gamma$ ($\theta(\psi)$ is a certain function of the stream function, γ is the adiabatic exponent) by the functions (14). The functions $p_2 = \theta(\psi)\rho_2^\gamma$, $p_2 = A(\psi) - B(\psi)/\rho_2$ in the $p_2\rho_2$-plane define one-parameter families of curves with parameter ψ. We shall require that, for the prescribed value ρ_{20}, each curve of one family be tangent to the corresponding curve of the other family.

Equating the functions and their first derivatives at ρ_{20}, one readily obtains

$$\chi^2 = \rho_1 \left[1 - a\rho_{20}^\gamma(\gamma + 1)\theta(\psi)\right] / 2a\rho_{20}^{\gamma+1}\gamma\theta(\psi),$$

$$i_1 = \left[2a(\gamma + 1)\rho_{20}^\gamma\theta(\psi) - 1\right] / 2a\rho_1 \left[1 - a\rho_{20}^\gamma(\gamma + 1)\theta(\psi)\right].$$

If it is assumed that in both planes the motion of the gas is irrotational, then, taking $\theta(\psi) = p_{20}/\rho_{20}^\gamma = \text{const}$, we obtain A and B constant—this is the well-known result of S. A. Chaplygin (2).

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