EXTREMAL PROBLEMS FOR SOME CLASSES OF ANALYTIC FUNCTIONS WITH BOUNDED MEAN MODULUS

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Abstract

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EXTREMAL PROBLEMS FOR SOME CLASSES OF ANALYTIC FUNCTIONS WITH BOUNDED MEAN MODULUS

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This paper is devoted to the study of extremal problems in classes of analytic functions generalizing the well-known classes H_δ, $\delta > 0$. Extremal problems in the classes H_δ with $\delta \geqslant 1$ have been studied by many authors. We mention here the papers ([1–5]), in which one can find the history of the question and an extensive bibliography. Extremal problems in the classes H_δ with $\delta < 1$ have been less studied. In paper ([2]) it is indicated that S. Ya. Khavinson found the form of the extremal function in the class H_δ, $0 < \delta < 1$, for the problem of

$$\sup \left| \sum_{j=1}^{\nu} \sum_{i=0}^{n_j} \gamma_{ij} f^{(i)}(z_j) \right|,$$

where $z_1, \dots, z_\nu$ are prescribed points of the unit disk; γ_{ij} are prescribed coefficients. However, this result was not published. In papers ([6],[7]) V. Kabaila found the form of the function with least norm

$$\|f\|_\delta = \sup_{0<r<1} \left\{ \int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta \right\}^{1/\delta},$$

interpolating prescribed values at prescribed points. In the same papers the problem of $\sup |f'(z_0)|$, $f \in H_\delta$, $0 < \delta < 1$, was solved. In a recent paper ([8]) S. A. Gel' fer and L. V. Kresnyakova considered a number of extremal problems in the classes H_δ, $\delta > 0$, but the form of the extremal functions found by them can in some cases be substantially refined.

Let $m(u) \geqslant 0$ be a convex nondecreasing function, defined for $-\infty < u < +\infty$, possessing a continuous derivative $p(u) = m'(u)$ and satisfying the conditions

$$\lim_{u \to -\infty} m(u) = 0, \qquad \lim_{u \to +\infty} m(u)/u = +\infty.$$

The function $p(u)$ has an inverse $q(t)$, continuous and increasing for $0 < t < +\infty$.

We now introduce the class H_m of functions $f(z)$, analytic in the unit disk, for which

$$\int_0^{2\pi} m[\ln|f(re^{i\theta})|]\,d\theta \leqslant 1. \tag{1}$$

Functions of the classes H_m were first considered in the work of E. D. Solomentsev ([9]) from the standpoint of their boundary behavior. From the results of ([9]) there follows a parametric representation of V. I. Smirnov type for functions from H_m:

$$f(z) = B(z)\exp\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\nu(\theta). \tag{2}$$

Here $B(z)$ is a Blaschke product, and the Borel measure $d\nu(\theta)$ has the following structure: $d\nu(\theta) = u(\theta)\,d\theta + d\mu(\theta)$, where $u(\theta)$ is a summable function satisfying the condition

$$\int_0^{2\pi} m[u(\theta)]\,d\theta \leqslant 1, \tag{3}$$

$d\mu \leq 0$ is a singular measure. If $m(u) \equiv e^{\delta u}$, $\delta > 0$, then the class H_m coincides with H_δ; if $m(u) \equiv M(e^u)$, where $M(t)$ is an N-function (see (10)), then one obtains the classes H_M, the extremal problems in which were considered by us in ([11,12]).

Let us introduce some further notation. By T_m we shall denote the class of Borel measures $d\nu(\theta)$ which occur in (2). By Q_m^* we shall denote the class of functions $\rho(z)$ having the representation

$$\rho(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\nu(\theta), \qquad \nu \in T_m, \tag{4}$$

and by Q_m the subclass of Q_m^* consisting of functions in whose representation, according to formula (4), the singular component of the measure $d\nu$ is equal to zero. By H_m^0 we shall denote the class of functions having the representation $\exp\rho(z)$, $\rho(z) \in Q_m^*$. If $z_1, \dots, z_\nu$ are points of the unit disk, $n_1, \dots, n_\nu$ are integers ≥ 1, $n = n_1 + \dots + n_\nu$, and $A = \{F(z)\}$ is some class of functions analytic in the unit disk, then by $W(A, \{z_i n_i\})$, or more briefly $W(A)$, we shall denote the set of points $(F(z_1), \dots, F^{(n_1-1)}(z_1), \dots, \dots, F(z_\nu), \dots, F^{n_\nu - 1}(z_\nu))$.

Lemma 1. Let $\alpha(\theta)$ be a continuous function on $[0, 2\pi]$. In order that the upper bound

$$\sup_{\nu \in T_m} \int_0^{2\pi} \alpha(\theta)\, d\nu$$

be finite, it is necessary and sufficient that $\alpha(\theta) \geq 0$.

Lemma 2. Let $\alpha(\theta) \geq 0$ be continuous and not identically zero on $[0, 2\pi]$. There exists a constant $k^* > 0$ such that

$$\int_0^{2\pi} m\{q[k^* \alpha(\theta)]\}\, d\theta = 1, \tag{5}$$

$$\sup_{\nu \in T_m} \int_0^{2\pi} \alpha(\theta)\, d\nu = \int_0^{2\pi} \alpha(\theta)\, q[k^* \alpha(\theta)]\, d\theta. \tag{6}$$

If, moreover, $q[k^* \alpha(\theta)]$ is an integrable function, then any measure $d\nu^*$ extremal for (6) has the form

$$d\nu^* = q[k^* \alpha(\theta)]\, d\theta + d\mu^*, \tag{7}$$

where the singular measure $d\mu^*$ is concentrated on the set of roots of the equation $\alpha(\theta) = 0$. Conversely, every measure of the form (7) will be extremal in (6). If $q(k^* \alpha(\theta))$ is not an integrable function, then the upper bound (6) in the class T_m is not attained.

In the following theorems the sets W defined above are considered in n-dimensional complex space C^n if $0 \in \{z_1, \dots, z_\nu\}$, or in $(2n-1)$-dimensional real space R^{2n-1} if $z_\nu = 0$ (in the latter case the coordinate $F(z_\nu)$ is always real for us).

Theorem 1. Each of the sets $W(Q_m)$ and $W(Q_m^*)$ is unbounded, convex, and contains interior points. The set $W(Q_m^*)$ is closed.

Let

$$\Phi(\theta, z) = \frac{e^{i\theta} + z}{e^{i\theta} - z}, \quad \alpha_1(\theta) = \Phi(\theta, z_1), \quad \alpha_2(\theta) = \left.\frac{\partial \Phi}{\partial z}\right|_{z=z_1}, \dots, \alpha_{n_1}(\theta) = \left.\frac{\partial^{n_1 - 1} \Phi}{\partial z^{n_1 - 1}}\right|_{z=z_1},$$

$$\alpha_{n_1+1}(\theta) = \Phi(\theta, z_2), \dots, \dots, \alpha_n(\theta) = \left.\frac{\partial^{n_\nu - 1} \Phi}{\partial z^{n_\nu - 1}}\right|_{z=z_\nu}.$$

We shall say that the measure $d\nu$ corresponds to the point $(\xi) = (\xi_1, \dots, \xi_n)$ if

$$\xi_j = \int_0^{2\pi} \alpha_j(\theta)\, d\nu, \qquad j = 1, \dots, n.$$

Theorem 2. To each boundary point $(\xi^*) = (\xi_1^*, \dots, \xi_n^*)$ of the body $W(Q_m^*)$ there corresponds a unique measure $d\nu^* \in T_m$ of the form:

$$d\nu^* = q[\alpha(\theta)\, d\theta] + d\mu^*, \tag{8}$$

where

$$\alpha(\theta) = \mathrm{Re} \sum_1^n \eta_k \alpha_k(\theta) \geq 0, \tag{9}$$

$$\int_0^{2\pi} m[q(\alpha(\theta))]\, d\theta = 1 \tag{10}$$

and the function $q[\alpha(\theta)]$ is summable, while the measure $d\mu^*$ is concentrated on the set of roots of the equation $\alpha(\theta) = 0$. The coefficients η_k are determined uniquely by the point $(\xi^*) = (\xi_1^*, \dots, \xi_n^*)$ under condition (10).

Conversely, to every measure of the form (8), where $\alpha(\theta)$ and $d\mu^*$ satisfy the conditions described and $q[\alpha(\theta)]$ is summable, there corresponds a boundary point (ξ^*) of the body $W(Q_m^*)$.

Lemma 3. Let r be the multiplicity of a root of $\alpha(\theta)$. In order that the function $q[\alpha(\theta)]$ be summable in a neighborhood of this root, it is necessary and sufficient that the integral

$$\int_{-\infty}^{\cdot} u\, [m'(u)]^{1/r-1} m''(u)\, du \tag{11}$$

converge.

If $m(u) = e^{\partial u}$, then the integral (11) converges for all $r \geq 2$. Hence the function $\alpha(\theta)$ in (9) may have roots of arbitrary multiplicity. If $m(u) = (-u)^{-h}$ ($h > 0$, $u < 0$), then the integral (11) converges only for $r < h + 1$. Therefore, for $0 < h < 1$, the function $\alpha(\theta)$ in (9) cannot have roots, and the singular component in (8) is absent. If $h > 1$, the admissible multiplicity r of a root is determined by the inequality $r < h + 1$.

Theorem 3. Through each boundary point $(\xi^*) \in W(Q_m^*)$ there passes only one hyperplane supporting $W(Q_m^*)$.

2. Each hyperplane supporting $W(Q_m^*)$ has one and only one point in common with $W(Q_m)$.

3. a) The closure $\overline{W(Q_m)}$ coincides with $W(Q_m^*)$.

b) The body $W(Q_m)$, generally speaking, is not closed. This will be the case, in particular, if the integral (11) converges for all $r \geq 2$.

c) If, however, the integral (11) diverges for every $r \geq 2$, then $W(Q_m)$ is closed and coincides with $W(Q_m^*)$.

Theorem 4. Suppose the function $m(u)$ is not linear on any interval. Then the following conditions are equivalent:

1. The point (ξ^*) is an extreme point of the body $W(Q_m^*)$.
2. In the representation of the measure $d\nu^*$ by formula (8), corresponding to the point (ξ^*), the singular component $d\mu$ is absent.
3. The point $(\xi^*) \in \partial W(Q_m^*) \cap W(Q_m)$, where $\partial W(Q_m^*)$ is the boundary of the body $W(Q_m^*)$.

Theorem 5 establishes the form of the extremal functions in the class Q_m^*.

Theorem 5. Let $0 \in \{z_1, \dots, z_\nu\}$. To each boundary point $(\xi^*) \in W(Q_m^*)$ there corresponds in Q_m^* a unique function $\rho^*(z)$ of the form

$$\rho^*(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, q\left[\frac{a \prod_1^n |1 - \overline{\gamma_j} e^{i\theta}|^2}{\prod_1^\nu |e^{i\theta} - z_j|^{2n_j}}\right] d\theta + \sum \mu_j \frac{z + \gamma_j}{z - \gamma_j}, \tag{12}$$

where $\gamma_1, \dots, \gamma_n$ are certain points in the disk $|z| \leq 1$, Σ extends over those γ_j for which $|\gamma_j| = 1$, all $\mu_j \leq 0$, and the constant a is chosen so that

$$\int_0^{2\pi} m\left\{q\left[\frac{a \prod_1^n |1 - \overline{\gamma_j} e^{i\theta}|^2}{\prod_1^\nu |e^{i\theta} - z_j|^{2n_j}}\right]\right\} d\theta = 1. \tag{13}$$

A factor of the form $|1 - \bar{\gamma}_j e^{i\theta}|$, $|\gamma_j| = 1$, may enter into the structure of $\rho(z)$ to the power r (r an even number ≥ 2) only in the case when the integral (11) converges. In the case $z_\nu = 0$, in (12) and (13) n must be replaced by $n-1$, and ν by $\nu - 1$.

Theorem 6. To boundary points of the set $W(H_m^0)$ there correspond in H_m^0 only functions of the form $\exp[\rho^*(z)]$, where the construction of $\rho^*(z)$ is described in Theorem 5.

Theorem 7. To boundary points of the set $W(H_m)$, if $0 \notin \{z_1, \dots, z_\nu\}$, there correspond only functions of the form

$$f^*(z) = e^{i\lambda} \prod_1^{n-1} \frac{z - \beta_j}{1 - \bar{\beta}_j z} \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, q\left[a\, \frac{\prod_1^n |1 - \bar{\gamma}_j e^{i\theta}|^2}{\prod_1^\nu |e^{i\theta} - z_j|^{2n_j}}\right] d\theta\right\}. \tag{14}$$

Here $|\beta_j| \le 1$, $j = 1, \dots, n-1$, are certain points, λ is a real number, and, with respect to $\{\gamma_j\}$, all the conditions of Theorem 5 are fulfilled. If $z_\nu = 0$, then under the sign exp one must replace n by $n-1$, and ν by $\nu - 1$.

For the class H_δ, $\delta > 0$, using the work of Kabaila ([6],[7]), it is not difficult to obtain a result more precise than the one following directly from Theorem 7.

Theorem 8. To boundary points of the set $W(H_\delta)$ there correspond only functions of the form

$$f^*(z) = ae^{i\lambda} \prod \frac{z-\gamma_j}{1-\bar{\gamma}_j z} \cdot \prod_1^{n-1} (1-\gamma_j z)^{2/\delta} \cdot \prod_1^{\nu} (1-\bar{z}_j z)^{-2n_j/\delta}, \qquad (15)$$

where $|\gamma_j| \le 1$, $j = 1, \dots, n-1$; $\prod$ extends over some of the $\{\gamma_j\}$; $a > 0$, λ are real constants; $\|f^*\|_\delta = 1$.

Corollary. Let the function $u(c_1, \dots, c_n)$ be continuous on $W(H_m)$ and unable to attain its supremum at interior points of this set. Then the extremal functions in the problem

$$\sup_{f \in H_m} u(f(z_1), \dots, f^{(n_1-1)}(z_1), \dots, f^{(n_\nu - 1)}(z_\nu))$$

have the form (14) $(m(u) \neq e^{\delta u})$ or (16) $(m(u) = e^{\delta u})$.

In the paper ([8]) the problem of $\sup \operatorname{Re} \Phi(f'(z))$, $f \in H_\delta$, $\delta > 0$, was considered, where $\Phi(t)$ is an entire function. The form of the extremal function indicated there is inaccurate, since, first, it is not deciphered what the Blaschke product will be, and, second, the factor with the singular component must be absent.

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Note: Figure translations are in progress. See original paper for figures.