ON METRIC AUTOMORPHISMS WITH SIMPLE SPECTRUM

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Abstract

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ON METRIC AUTOMORPHISMS WITH SIMPLE SPECTRUM

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Let M be the unit interval with the usual Lebesgue measure μ; let $\mathfrak{A}$ be the set of all automorphisms (invertible measure-preserving transformations) of the space M, endowed with the weak topology (see ([2])), and let $\mathfrak{P}$ be the set of automorphisms in $\mathfrak{A}$ having simple spectrum (see ([1])). The purpose of the present note is to prove the following theorem.

Theorem 1. *In the space $\mathfrak{A}$, the complement of the set $\mathfrak{P}$ is of first category.*

From this theorem and from the well-known theorem on the massiveness of the set of automorphisms with continuous spectrum in $\mathfrak{A}$ (see ([1])) it follows that in $\mathfrak{A}$ there exists an everywhere dense G_δ consisting of automorphisms with simple continuous spectrum.

History of the question. Even before the war A. N. Kolmogorov indicated an example of an automorphism for which one could suppose that it has simple spectrum. A proof of the simplicity of the spectrum in this example was found in 1958 by I. V. Girsanov ([3]). This was an example of a probability-theoretic origin. In September 1965, at one of the sessions of the Khumsan school on ergodic theory, A. B. Katok, A. G. Kushnirenko, V. I. Oseledets, and A. M. Stepin presented simpler and more geometric examples of automorphisms with simple and finite-multiple nondiscrete spectrum, together with methods for estimating the multiplicity of the spectrum. In November 1965, at his seminar on ergodic theory, V. A. Rokhlin expressed the supposition that a suitable generalization of these methods, together with more old-fashioned tools, would make it possible to prove Theorem 1. The present work is an implementation of this program.

2. We shall call **dyadic intervals of rank** n $(n = 1, 2, \dots)$ the intervals $C_n^i = (i/2^n, (i+1)/2^n)$ $(i = 0, 1, \dots, 2^n - 1)$, and **dyadic permutations of rank** n automorphisms of the space M cyclically permuting these intervals. For an arbitrary positive sequence $f(n)$, denote by $\mathfrak{B}_f$ the set of automorphisms $T \in \mathfrak{A}$ such that $T \in \mathfrak{B}_f$ if and only if, for some strictly increasing sequence of natural numbers q_n and a sequence of dyadic permutations S_n of rank q_n,

$$\mu(TC^i_{q_n}\Delta S_n C^i_{q_n}) < f(q_n), \qquad i = 0, 1, \dots, 2^{q_n} - 1; \quad n = 1, 2, \dots$$

(Δ denotes the symmetric difference).

We shall prove that *for any sequence f the set $\mathfrak{B}_f$ is an everywhere dense G_δ in $\mathfrak{A}$.* To this end denote by $\mathfrak{B}^m_f$ the set of automorphisms $T \in \mathfrak{A}$ such that $T \in \mathfrak{B}^m_f$ if and only if, for some dyadic permutation S_m of rank m,

$$\mu(TC^i_m \Delta S_m C^i_m) < f(m), \qquad i = 0, 1, \dots, 2^m - 1.$$

It is clear that the set $\mathfrak{B}^m_f$ is open in $\mathfrak{A}$ and

$$\mathfrak{B}_f = \bigcap_{l=1}^{\infty} \bigcup_{m=l}^{\infty} \mathfrak{B}^m_f.$$

Consequently, $\mathfrak{B}_f$ is G_δ. Next, let α be an irrational number for which there exists a sequence of irreducible fractions $\alpha_n = p_n/2^{k_n}$ with strictly increasing denominators such that $|\alpha - \alpha_n| < \frac{1}{2} f(k_n)$. Denote by A the shift mod 1 by α, and by A_n the shift mod 1 by α_n. If Q is an arbitrary dyadic permutation of some rank s and n is such that $k_n > s$, then $QC^i_{k_n}$ $(i = 0, \dots, 2^{k_n} - 1)$ is a dyadic interval of rank k_n, the automorphism $Q^{-1}A_{nQ}$ is a dyadic permutation of rank k_n, and

$$\mu(Q^{-1}AQC^i_{k_n} \Delta Q^{-1}A_n QC^i_{k_n}) < f(k_n), \qquad i = 0, 1, \dots, 2^{k_n} - 1.$$

Consequently, $Q^{-1}AQ \in \mathfrak{B}_f$. Since in the space $\mathfrak{A}$ the set of all automorphisms conjugate to the automorphism A is everywhere dense ((2), p. 108), and the set of dyadic permutations is also everywhere dense ((2), p. 92), it follows that the set $\mathfrak{B}_f$ is everywhere dense.

3. We now show that if, for the sequence f, the condition

$$f(n) = o[2^{-n(5+2^n)}],$$

is satisfied, then $\mathfrak{B}_f \subset \mathfrak{P}$.

Let $T \in \mathfrak{B}_f$ and let $\{q_n\}, \{S_n\}$ be the corresponding sequences. Suppose, moreover, that β is a real number $(0 < \beta < 1)$ for which there exists a sequence of irreducible rational fractions $\beta_n = t_n/r_n$, where $r_n = 2^{q_{k_n}}$, $k_n \to \infty$, such that

$$|\beta - \beta_n| < (r_n)^{-(r_n+4)}. \tag{1}$$

Put $R_n = S_{k_n}$, and denote by U the operator in $L^2(M)$ conjugate to T, by V_n the operator conjugate to R_n, and by χ, χ_n, and $\tilde{\chi}_n$ the characteristic functions of the intervals $[0,\beta)$, $[0,\beta_n)$, and $[0,1/r_n)$. We shall prove that the function χ is a generating element for U, i.e. that the linear span of the set $\{U^i\chi\}$ $(i=0,1,...)$ is everywhere dense in $L^2(M)$.

First note that for any n the function χ_n is a generating element for V_n in the subspace L_n of functions that are constant on dyadic intervals of rank q_{k_n}. Indeed, otherwise in L_n there would be an eigenfunction of the operator V_n orthogonal to χ_n, i.e., in other words, the equality $\sum_{i=1}^{p}\lambda_i = 0$ would hold, where λ_i are roots of unity of degree r_n and p is an odd number $(1 \le p < r_n)$. But it is not difficult to show by induction on n that this equality is impossible.

Now let g be a bounded function from $L^2(M)$, $|g(x)| < K$ $(x \in M)$, and let ε be an arbitrary positive number. Since $r_n \to \infty$ and R_n cyclically permutes dyadic intervals of rank q_{k_n}, there exists a natural number N such that for $n > N$

$$\left\| g - \sum_{i=1}^{r_n} c_i V_n^i \tilde{\chi}_n \right\| < \varepsilon, \tag{2}$$

where c_i are some real numbers depending on n, and $\sum |c_i| < K$. Fix some number $n > N$ for which

$$2K\sqrt{f(q_{k_n})}\,(r_n)^{(r_n+5)/2} < \varepsilon, \tag{3}$$

$$2K/r_n < \varepsilon. \tag{4}$$

Since $\tilde{\chi}_n$ is a generating element for V_n in L_n, in particular,

$$\tilde{\chi}_n = \sum_j d_j V_n^j \chi_n,$$

where, according to Hadamard' s inequality, $|d_j| < (r_n)^{r_n/2}$. Thus–

it follows that

$$\left\| g - \sum_i c_i V_n^i \tilde{\chi}_n \right\| = \left\| g - \sum_i c_i V_n^i \chi_n \right\|, \tag{5}$$

where

$$\tilde{c}_i = \sum_{\substack{k+l=i \\ (\mathrm{mod}\, m)}} c_k d_l,$$

and therefore

$$|\tilde{c}_i| < 2K(r_n)r_n/2. \tag{6}$$

Applying the triangle inequality, we obtain

$$\left\| g - \sum_i \tilde{c}_i U^i \chi \right\| \leq \left\| g - \sum_i \tilde{c}_i V_n^i \chi_n \right\| + \sum_i |\tilde{c}_i| \, \|V_n^i \chi_n - U^i \chi_n\| + \sum_i |\tilde{c}_i| \, \|\chi_n - \chi\|.$$

We estimate each of the three terms on the right-hand side. According to (2) and (5),

$$\left\| g - \sum_i \tilde{c}_i V_n^i \chi_n \right\| < \varepsilon. \tag{7}$$

Put

$$A = \sum_i |\tilde{c}_i| \, \|V_n^i \chi_n - U^i \chi_n\|$$

and

$$B = \sum_i |\tilde{c}_i| \, \|\chi_n - \chi\|.$$

Since

$$A \leq \sum_i |\tilde{c}_i| \sum_{j=0}^{i-1} \|V_n V_n^j \chi_n - U V_n^j \chi_n\| = \sum_i |\tilde{c}_i| \sum_{j=0}^{i-1} \sqrt{\mu(R_n E_n^j \Delta T E_n^j)},$$

where E_n^j are certain subsets of the space M consisting of dyadic intervals of rank q_{k_n}, it follows from (6) and the properties of the sequence $\{R_n\}$ that

$$A \leq 2K(r_n)(r_n + 5)/2\sqrt{f(q_{k_n})},$$

and, according to (3),

$$A < \varepsilon. \tag{8}$$

Finally,

$$B = \sqrt{|\beta - \beta_n|} \sum_i |\tilde{c}_i|,$$

and from (1), (6), and (4) it follows that

$$B < 2K/r_n < \varepsilon. \tag{9}$$

From (7), (8), and (9) it follows that

$$\left\| g - \sum_i \tilde{c}_i U^i \chi \right\| < 3\varepsilon. \tag{10}$$

In view of the arbitrariness of the choice of ε and g, inequality (10) means that the linear span of the set $\{U^i \chi\}$ $(i = 0, 1, ...)$ is everywhere dense in the set of bounded functions in $L^2(M)$, and consequently everywhere dense in $L^2(M)$.

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Note: Figure translations are in progress. See original paper for figures.