ON A CLASS OF FINITELY PRESENTED GROUPS

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Abstract

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MATHEMATICS

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ON A CLASS OF FINITELY PRESENTED GROUPS

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We shall denote by K the class of all finitely presented groups $G = \langle a_1, \dots, a_n;\ R_1 = 1, \dots, R_k = 1\rangle$ such that: 1) the set of all defining words $\{R_i\}$ is closed under the operation of taking inverses and cyclic permutation of letters; 2) each R_i is cyclically irreducible; 3) when reducing the product of two words R_{j_1} and R_{j_2} from $\{R_i\}$ that are not mutually inverse, less than 1/4 of the letters of the word R_j are cancelled; 4) if the defining words $R_{i_1}, R_{i_2}, R_{i_3}$ are written on a circle one after another and a reduction occurs between each pair, then less than 1/4 of each of the words R_{i_q}, $q = 1, 2, 3$, is cancelled.

For groups of the class K, the identity problem is solved by Dehn' s algorithm, and the conjugacy problem by the generalized Dehn algorithm.

In the work ([1]), M. Grindlinger gave positive solutions of the identity and conjugacy problems for a certain class of groups narrower than K, and in the work ([3]) Lyndon did so for certain other classes, which partially intersect the class K.

We shall use the following notation. If A is a word written in the generators of the group G, then the inverse word to it will be denoted by $\overline{A}$. By $\equiv$, $\cong$, $=$ we shall denote, respectively, free equality, graphical equality, and equality in the group G.

1°. In every group of the class K, every word equal to the identity is freely equal to a product of n factors of the form $\overline{T}_i R_i T_i$, where R_i is some defining word; T_i contains no more than half of some defining word, $\overline{T}_i R_i T_i$ is irreducible; for any R_j and R_k from

$$\prod_{i=1}^{n} \overline{T}_i R_i T_i$$

such that $R_j \cong XYZ,\ R_k \cong P\overline{Y}Q$, and under some way of reducing this product the words Y and $\overline{Y}$ cancel with one another, we have $ZXY \ncong \overline{Y}QP$.

The product $R''_{i_1} \dots R''_{i_p}$, remaining under some way of reducing

$$\prod_{i=1}^{n} \overline{T}_i R_i T_i,$$

we shall call an **adjacent product** if

$$\text{for } p = 1 \quad l(R''_{i_1}) > {}^3/_4\, l(R_{i_1}),$$
$$\text{for } p > 1 \quad l(R''_{i_q}) > {}^1/_2\, l(R_{i_q}), \quad q = 1, p,$$
$$l(R''_{i_j}) > {}^1/_4\, l(R_{i_j}), \quad j = 2, \dots, p-1,$$

each R_{i_q} is reduced with $R_{i_{q+1}}$, $q = 1, \dots, p-1$, and no R_{i_s} is reduced with any $\overline{T}_{i_q}$ or T_{i_r}.

Theorem 1. *If a nonempty irreducible word C is equal to the identity in a group G from K, then under any way of reducing the product*

$$\prod_{i=1}^{n} \overline{T}_i R_i T_i \equiv C,$$

the word C contains an adjacent product.

It is easy to see that Theorem 1 implies a positive solution of the identity problem for groups of the class K.

2°. For every group in K, denote by $\mathcal{A}_1$ the set of all defining words R_i; of the parts of the defining words R'_i remaining after reduction of the product of two not mutually inverse defining words; and of the parts R''_i of defining words remaining after reduction of R_i, R_j, R_k, when they are written one after another on a circle and a reduction occurs between each pair.

Let the sets $\mathcal{A}_2, \dots, \mathcal{A}_{n-1}$ be defined. By $\mathcal{A}_n$ we shall denote the set of all words freely equal to a product XY such that

$$X \cong R^*_{i_1} \dots R^*_{i_{n-1}} \in \mathcal{A}_{n-1}, \quad Y \cong R^*_j \in \mathcal{A}_1,$$

$R^*_{i_{n-1}}$ and R^*_j are not parts of mutually inverse defining words, and $R^*_{i_{n-1}} R^*_j$ is reducible.

The set

$$\bigcup_{n=1}^{\infty} \mathcal{A}_n$$

will be denoted by $\mathcal{A}$.

Lemma. *A nonempty irreducible cyclic word equal to 1, not a defining word, and folded into a cycle, contains an occurrence of two nonintersecting words from the set $\mathcal{A}$.*

Let words A and B be given. In order to determine whether these words are conjugate or not, we fold each of them into a cycle, reduce them, and, if it contains a word S whose length is more than half of a defining word $R \cong ST$, replace it by the smaller part $\overline{T}$. If the word A transformed in this way is nonempty, we consider all its cyclic permutations $A_1, \dots, A_s$. If there exists a defining word $R_i \cong XA_i\overline{X}D$ and $l(D) < {}^1/_2 l(R_i)$, then we add to the sequence $A_1, \dots, A_s$ all possible permutations of D. We proceed analogously with the word B. We obtain two sequences $A_1, \dots, A_\alpha$ and $B_1, \dots, B_\beta$.

Theorem 2. *The words A and B are conjugate if and only if at least one of the relations $A_i = \overline{X}B_jX$ holds, where $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$, and $l(X) \leq {}^1/_2 \max\{l(R_i)\}$.*

3°. In the following theorems a number of abstract properties of groups of the class K are established. In the proofs of these theorems the lemma given above is used.

Theorem 3. *In every group of K which is not cyclic, no nontrivial identity holds.*

Theorem 4. *Every noncyclic group of K is a group without a center.*

Theorem 5. *Every element of a group in K has trivial centralizer.*

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REFERENCES

1. M. Grindlinger, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **29**, No. 2, 245 (1965).
2. M. Grindlinger, *DAN*, **154**, No. 3, 507 (1964).
3. R. C. Lyndon, *Notices Am. Math. Soc.*, **12**, No. 3, 370 (1965).

Note: Figure translations are in progress. See original paper for figures.