CASIMIR OPERATORS FOR CLASSICAL GROUPS

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 519.46

MATHEMATICS

A. M. PERELOMOV, V. S. POPOV

CASIMIR OPERATORS FOR CLASSICAL GROUPS

(Presented by Academician I. G. Petrovskii, 7 VII 1966)

Invariant operators constructed from the generators of a group (Casimir operators ([1])) are important both for group theory itself and for applications. These operators were first considered in general form by I. M. Gelfand ([2]) and Racah ([3]). Although an extensive literature has been devoted to their properties (see, for example, ([2–7])), explicit formulas for the eigenvalues of Casimir operators of arbitrary order had not previously been obtained. Below we give a solution of this problem for all "classical" groups (in the terminology of H. Weyl).

Let us first consider the groups $U(n)$ and $SU(n)$. Their generators satisfy the commutation relations

$$[A^i_j,\ A^k_l] = \delta^k_j A^i_l - \delta^i_l A^k_j. \tag{1}$$

The Casimir operator of order p has the form

$$C_p = \sum_{i_1,\dots,i_p} A^{i_1}_{i_2} A^{i_2}_{i_3} \dots A^{i_p}_{i_1}. \tag{2}$$

An irreducible representation is specified by the Young diagram $\{f_1, f_2, \dots, f_n\}$, $f_1 \geq f_2 \geq \dots \geq f_n \geq 0$; here f_i is the number of boxes in the i-th row. Since the operator C_p in any irreducible representation reduces to a constant, to find its eigenvalue $C_p(f_1, f_2, \dots, f_n)$ we apply C_p to the highest vector of the representation, satisfying the relations

$$A^i_i \psi_0 = m\psi, \qquad A^i_j \psi = 0 \quad \text{for } i < j. \tag{3}$$

The numbers m_i are the components of the highest weight of the irreducible representation;

$m_i = f_i$ for the group $U(n)$, $\qquad m_i = f_i - \frac{1}{n}(f_1 + f_2 + \dots + f_n)$ for the group $SU(n)$.

Rewriting C_p in the form

$$C_p = \sum_{i,j=1}^{n} (T_{p-1})^i_j A^j_i, \qquad \text{where} \ \ (T_{p-1})^i_j = \sum_{i_1,\dots,i_{p-2}=1}^{n} A^i_{i_1} A^{i_1}_{i_2} \dots A^{i_{p-2}}_j, \quad (4)$$

we note that the operator $(T_{p-1})^i_j$ has the same transformation properties as the generator A^i_j. Hence

$$C_p \psi_0 = \sum_{i=1}^{n} (m_i + 2r_i)(T_{p-1})^i_i \psi, \quad (5)$$

$$(T_q)^i_i \psi = \sum_{j=1}^{n} a_{ij} (T_{q-1})^j_j \psi. \quad (6)$$

Here $r_i = (n+1)/2 - i$ (the half-sum of the positive roots of the unitary group),

$$a_{ij} = (m_i + n - i)\delta_{ij} - \theta_{ij}, \qquad \theta_{ij} = \begin{cases} 1, & \text{for } i < j, \\ 0, & \text{for } i \geq j. \end{cases} \quad (7)$$

Proceeding recursively, we arrive at the final result ([8],[9])

$$C_p(f_1, f_2, \dots, f_n) = \sum_{i,j} (a^p)_{ij} = \mathrm{Sp}(a^p E), \quad (8)$$

where the matrix E, consisting entirely of ones, has been introduced: $E_{ij} = 1$ for all i, j. If the matrix a is brought to diagonal form, then we obtain the following expression for C_p:

$$C_p(f_1, f_2, \dots, f_n) = \sum_{i=1}^{n} \lambda_i^p \prod_{j \neq i} \frac{\lambda_i - \lambda_j - 1}{\lambda_i - \lambda_j}, \qquad \lambda_i = m_i + n - i. \quad (9)$$

Formula (9) represents $C_p(f_1, f_2, \dots, f_n)$ in a uniform way for all values of p.* Its drawback, however, is that the individual terms in (9) are rational functions of the variables λ_i, whereas $C_p(f_1, f_2, \dots, f_n)$ as a whole reduces to a polynomial. For concrete computations it is convenient to expand C_p in a series in the power sums

$$S_k = \sum_{i=1}^{n} (\lambda_i^k - \rho_i^k),$$

$\rho_i = n - i$ (for the identity representation all S_k vanish). To obtain such an expansion, observe that (9) can be rewritten in the form of a contour integral

$$C_p(f_1, f_2, \dots, f_n) = -\frac{1}{2\pi i} \oint_{(+)} \lambda^p \prod_{i=1}^{n} \left(1 - \frac{1}{\lambda - \lambda_i}\right) d\lambda, \tag{10}$$

from which the form of the generating function for the Casimir operators follows immediately:

$$G(z) = \sum_{p=0}^{\infty} C_p z^p = n e^{-\varphi(z)} + \frac{1 - e^{-\varphi(z)}}{z}, \tag{11}$$

where

$$\varphi(z) = \sum_{k=2}^{\infty} a_k z^k, \qquad a_k = \sum_{l=1}^{k-1} \frac{(k-1)!}{l!(k-l)!} S_l. \tag{12}$$

Introduce the quantities B_p, defined by the expansion

$$e^{-\varphi(z)} = -\sum_{p=0}^{\infty} B_{p-1} z^p, \tag{13}$$

$$B_{-1} = -1, \quad B_0 = 0, \quad B_1 = S_1, \quad B_2 = S_2 + S_1, \quad B_3 = S_3 + \frac{3}{2} S_2 - \frac{1}{2} S_1^2 + S_1,$$

$$B_4 = S_4 + 2S_3 - S_2 S_1 + 2S_2 - S_1^2 + S_1, \dots$$

and so on. Then the final formula for the eigenvalues of the Casimir operators is written in the form

$$C_p(f_1, f_2, \dots, f_n) = B_p - n B_{p-1}. \tag{14}$$

The computations are carried out analogously for the remaining classical groups. The Casimir operator is still defined by formula (2); acting with it on the highest vector of an irreducible representation specified by the highest weight $\mathbf{m} = (m_1, m_2, \dots, m_n)$, $m_1 \geq m_2 \geq \dots \geq m_n$, for the eigenvalue $C_p(\mathbf{m})$ we obtain** formula (8), into which one must substitute the appropriate matrix a_{ij} for each of the groups:

$$a_{ij} = (l_i + \alpha)\delta_{ij} - \theta_{ij} + \beta \frac{1 + \varepsilon_i}{2} \delta_{i,-j}, \tag{15}$$

* The importance of representing C_p in the form (9) was pointed out to us by I. M. Gelfand.

** A more detailed presentation of the computations can be found in ([9]).

where $l_i = m_i + r_i$ $(i > 0)$*, $l_{-i} = -l_i$; the quantities $\alpha,\ \beta,\ r_i$ are given in Table 1, θ_{ij} is an upper triangular matrix all of whose elements standing above the main diagonal are equal to one (with $\theta_{ii} = 0$ for any i), and the quantities ε_i are equal to

Table 1

Group: Cartan designation	Group: other designations	Invariant form	α	β	r_i	Index i runs through the values	Order of the matrix a_{ij}
A_{n-1}	$SU(n)$	$\sum_{i=1}^{n} \bar{x}^i y^i$	$\frac{n-1}{2}$	0	$\frac{n+1}{2} - i$	$1, 2, \dots, n$	n
B_n	$O(2n+1)$	$x^0 y^0 + \sum_{i=1}^{n} \left(x^{i} y^{-i} + x^{-i} y^i\right)$	$n - \frac{1}{2}$	1	$\left(n + \frac{1}{2}\right)\varepsilon_i - i$	$1, 2, \dots, n, 0, -n, \dots, -2, -1$	$2n+1$
C_n	Sp$(2n)$	$\sum_{i=1}^{n} \left(x^{i} y^{-i} - x^{-i} y^i\right)$	n	-1	$(n+1)\varepsilon_i - i$	$1, 2, \dots, n, -n, \dots, -2, -1$	$2n$
D_n	$O(2n)$	$\sum_{i=1}^{n} \left(x^{i} y^{-i} + x^{-i} y^i\right)$	$n-1$	1	$n\varepsilon_i - i$	$1, 2, \dots, n, -n, \dots, -2, -1$	$2n$

$$\varepsilon_i = \begin{cases} 1 & \text{for } i > 0, \\ 0 & \text{for } i = 0, \\ -1 & \text{for } i < 0. \end{cases} \tag{16}$$

Formulas (8), (15) reduce the computation of the Casimir operators to the elementary problem of raising the known matrix a_{ij} to the p-th power. For small values of p the authors found ([8,9]) more convenient expressions in terms of the power sums S_k.

Acting by the same method as in the case of the unitary group, one can bring C_p to a form analogous to (9). These formulas, as well as expressions for the generating function of the Casimir operators for the orthogonal and symplectic groups, will be considered in the authors' next work.

As is known, in a group of rank n there are exactly n independent Casimir operators. It can be shown ([7,9]) that these are $C_1, C_2, \dots, C_n$ for the group $U(n)$ and $C_2, C_4, \dots, C_{2n}$ for the groups B_n, C_n, and D_n. For all classical groups except $O(2n)$, the listed Casimir operators uniquely determine the irreducible representation. The peculiarity of the group $O(2n)$ is that ([10]) it has two inequivalent spinor representations Δ_+ and Δ_-, whose highest weights differ only in the sign of the last component: $m_\pm = (1/2, \dots, 1/2, \pm 1/2)$. The eigenvalues of all operators C_{2k} on the representations Δ_+ and Δ_- coincide with one another; therefore, in order to restore a one-to-one correspondence between the weight vector $\mathbf{m}$ and the set of Casimir operators, one has to introduce a scalar operator $\tilde{C}$, different from (2). Such an operator for the group $O(2n)$ is the pseudoscalar

$$\tilde{C} = \sum_{(i,j)} \varepsilon^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_n} A^{i_1}_{j_1} A^{i_2}_{j_2} \cdots A^{i_n}_{j_n}, \tag{17}$$

analogous to the pseudoscalar $G = \frac{1}{8} i \varepsilon_{\mu\nu\sigma\rho} M^{\mu\nu} M^{\sigma\rho}$ for the Lorentz group. Its eigenvalues are equal to ([9])

$$\tilde{C}(\mathbf{m}) = (-1)^{n(n-1)/2} 2^n n!\, l_1 l_2 \cdots l_n. \tag{18}$$

* Here r_i is the half-sum of the positive roots of the group; note that for all the groups considered $r_i = (\alpha + 1)\varepsilon_i - i$.

In conclusion, we would like to draw the attention of mathematicians to the following unsolved problems:

1) Do formulas (8), (15) generalize to the case of exceptional groups (of type G_2, F_4, etc.), and also to the case of infinite-dimensional unitary representations of classical groups?

2) Along with the operators C_p defined in (2), symmetrized Casimir operators I_p occur in applications:

$$I_p = \frac{1}{p!} P \left(A^{i_1}_{i_2} \cdots A^{i_p}_{i_1} \right), \tag{19}$$

where the symbol P (the symmetrizer) denotes summation over all permutations of the generators standing in parentheses. The operator I_p is expressed in terms of C_q with $q \leq p$; the simplest formulas of this kind (for $p \leq 5$) were obtained in [6,9]. It is desirable to find the relation between I_p and C_p in general form, and also to determine whether the eigenvalue of I_p can be represented in a form analogous to (8) or (9).

The authors express their sincere gratitude to I. M. Gel' fand for discussion of the results of the present work.

Received
30 VI 1966

REFERENCES

[1] H. Casimir, B. L. Van der Waerden, Math. Ann., **111**, 1 (1935).
[2] I. M. Gel' fand, Matem. sborn., **26**, 103 (1950).
[3] G. Racah, Rend. Lincei, **8**, 108 (1950); Group Theory and Spectroscopy, Lecture notes, Princeton, 1951.
[4] L. C. Biedenharn, J. Math. Phys., **4**, 436 (1963).
[5] G. E. Baird, L. C. Biedenharn, J. Math. Phys., **4**, 1449 (1963).
[6] M. Umezawa, Nucl. Phys., **48**, 111 (1963); **53**, 54; **57**, 65 (1964).
[7] M. Micu, Nucl. Phys., **60**, 353 (1964).
[8] A. M. Perelomov, V. S. Popov, JETP, Letters to the Editor, **1**, no. 6, 15; **2**, no. 1, 34 (1965).
[9] A. M. Perelomov, V. S. Popov, Preprints of the Institute of Theoretical and Experimental Physics, Nos. 338, 352, Moscow, 1965.
[10] D. P. Zhelobenko, UMN, **17**, 27 (1962).

Note: Figure translations are in progress. See original paper for figures.