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1967

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Abstract

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UDC 519.46

MATHEMATICS

I. L. KANTOR

NONLINEAR GROUPS OF TRANSFORMATIONS DEFINED BY GENERAL NORMS OF JORDAN ALGEBRAS

(Presented by Academician I. G. Petrovskii, April 4, 1966)

In this note we study groups of transformations associated with the following invariants. Consider, in an n-dimensional real or complex vector space T_0, a nondegenerate* form $N(x, x, \dots, x) \equiv N(x)$ of degree k. To every four vectors x, y, a, b, given in a definite order, we assign the number

$$\frac{N(x-a)}{N(y-a)} : \frac{N(x-b)}{N(y-b)} \tag{1}$$

the double ratio of these vectors. To every k lines with direction vectors $p_1, p_2, \dots, p_k$ we assign the number

$$\frac{N(p_1, p_2, \dots, p_k)}{N(p_1)N(p_2)\cdots N(p_k)}. \tag{2}$$

In what follows we shall call a system of k lines a k-angle, and the number (2) the measure of the k-angle. Let us note that if $N(x)$ is a positive definite quadratic form, then the measure of a 2-angle is the square of the cosine of any one of the corresponding Euclidean angles.

Denote by G_1 the Lie algebra of infinitesimal transformations defined in a neighborhood of the origin O and preserving the double ratio (1) of any four vectors, and by G_2 the Lie algebra of infinitesimal transformations preserving the measure of k-angles in the differential-geometric sense. The latter means that k-angles formed by tangents to any k curves issuing from one point are considered.

We shall specify an infinitesimal transformation of the space T_0 in the form of the transformation $x' = x + F(x)t$, considered with accuracy up to infinitesimals of second order with respect to t. Here it is assumed that the vector-function $F(x)$ is represented, in some neighborhood of the origin O, in the form of a convergent series

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$$F(x) = a + A_1(x) + A_2(x,x) + \cdots + A_\nu(x,x,\ldots,x) + \cdots,$$

where a is a vector of the space T_0, $A_1(x)$ is a linear operator, and so on; $A_i(x,x,\ldots,x)$ is an operator of order i, acting in the space T_0 with values in T_0. In what follows infinitesimal transformations will be denoted simply by the corresponding functions $F(x)$. We shall need some definitions from ([3],[4]). Denote by T_i the space of all operators of order i acting in T_0 with values in T_0.

An algebra G of infinitesimal transformations is called the Lie algebra of a transitive differential group of order ν if the following conditions are fulfilled: 1) $G \supset T_0$; 2) $G \subset \sum_{i=0}^{\nu} T_i$, but $G \not\subset \sum_{i=0}^{\nu-1} T_i$; 3) if $A \in G$ and $A = A_0 + A_1 + \cdots + A_\nu$, where $A_i \in T_i$, then $A_i \in G$ $(i = 1,2,\ldots,\nu)$; 4) G is maximal, i.e. is not contained in any

* A k-linear form $N(a_1,a_2,\ldots,a_k)$ is called nondegenerate if from the fact that $N(a_1,a_2,\ldots,a_{k-1},x) = 0$ for arbitrary $a_1,a_2,\ldots,a_{k-1}$ it follows that $x = 0$.

any other Lie algebra of transformations satisfying conditions 1), 2), 3). We shall call an algebra of transformations satisfying conditions 1), 2), 3) an **algebra of the family** D^ν. An algebra of transformations containing at least one infinite series, satisfying condition 1) and satisfying the condition: 3′) if $A \in G$ and

$$A = A_0 + A_1 + \cdots + A_\nu + \cdots,$$

where $A_i \in T_i$, then $A_i \in G$ $(i = 1,2,\ldots)$, will be called an **algebra of the family** D^∞.

Theorem 1. *The Lie algebra G_1 of infinitesimal transformations preserving the double relation (1), where $N(x)$ is a nondegenerate form of degree k, is a Lie algebra of the family D^ν, where $\nu \le k-1$. The operators $A_i(x,x,\ldots,x)$ belonging to G_1 satisfy the relations (necessary and sufficient):*

$$N(x,x,\ldots,x,A_1(x)) = \lambda N(x,x,\ldots,x),$$

$$N(x,x,\ldots,x,A_2(a,x)) = f(a)N(x,x,\ldots,x),$$

$$N(x,x,\ldots,x,A_i(a_1,a_2,\ldots,a_{i-1},x)) = 0 \quad (i = 3,4,\ldots,k-1), \tag{3}$$

where λ is a number, and $f(a)$ is a linear function of the vector a.

Theorem 2. *The Lie algebra G_2 of infinitesimal transformations preserving the measure (2) of k-tuples, where $N(x)$ is a nondegenerate form of degree k, is a Lie algebra of the family D^ν, where $\nu \le \infty$. The operators belonging to G_2 satisfy the relations (necessary and sufficient)*

$$N(x,x,\ldots,x,A_1(x)) = \lambda N(x,x,\ldots,x),$$

$$N(x,x,\ldots,x,A_i(a_1,a_2,\ldots,a_{i-1},x)) = f_{i-1}(a_1,a_2,\ldots,a_{i-1})N(x) \quad (i = 2,3,\ldots), \tag{4}$$

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where λ is a number, and $f_{i-1}(a_1, a_2, \dots, a_{i-1})$ is an $(i-1)$-linear form in the vectors $a_1, a_2, \dots, a_{i-1}$.

It follows from Theorems 1 and 2 that the Lie algebra G_2, generally speaking, is broader than the Lie algebra G_1 and may even be infinite-dimensional (for example, in the case of conformal transformations of the Euclidean plane). However, the following holds.

Theorem 3. *If the Lie algebra G_2 is finite-dimensional, then it coincides with the Lie algebra G_1.*

It is clear that the principal interest is in those nondegenerate forms $N(x, x, \dots, x)$ for which the Lie algebras G_1 and G_2 contain operators of order higher than the first. Then, as can be shown using Theorems 1 and 2, there must exist a bilinear symmetric operator $A(x, y)$ for which the equality

$$N(x, x, \dots, x, A(a, x)) = f(a)N(x, x, \dots, x), \tag{5}$$

holds, where $f(a)$ is a linear form in the vector a.

Theorem 4. *Suppose that for the nondegenerate form $N(x, x, \dots, x)$ there exists a bilinear symmetric operator $A(x, y)$ such that relation (5) holds, and there exists a vector e such that for every x one has $A(e, x) = x$. Then:*

1) *the bilinear operator $A(x, y)$ turns T_0 into a semisimple Jordan algebra S with multiplication $xy = A(x, y)$;*

2) *the irreducible factors of the form $N(x, x, \dots, x)$ coincide with the irreducible factors of the generic norm of the algebra S;*

3) *the Lie algebra G_1 is the Lie algebra of the transitive differential group generated by the algebra S;*

4) *the Lie algebra G_2 does not coincide with the Lie algebra G_1 in precisely those cases when the Jordan algebra S contains: a) in the complex case—one-dimensional ideals; b) in the real case—one-dimensional or two-dimensional ideals.*

The notion of the generic norm of a Jordan algebra was introduced by N. Jacobson in the papers (1, 2). There the Lie algebra of infinitesimal

small linear transformations preserving the general norm. The Lie algebra G_1 defined in the theorem contains this algebra as its part. The Lie algebra G_1 includes all vectors, all linear operators of the form $T_a(x) \equiv ax$ and $[T_a, T_b]$, and all quadratic operators of the form $ax^2 - 2(ax)x$, where multiplication of vectors is performed in the sense of the semisimple Jordan algebra S (see ([3],[4])).

Let us pass to the definition of the corresponding transformation groups. It can be proved that every (finite) transformation of the space T_0 preserving the double ratio (1) or the measure of k-angles (2), and defined in some neighborhood, is analytic and extends uniquely to the whole space T_0, with the exception of the points of a set of measure zero. For any two such transformations their

product is naturally defined. Therefore the group $\mathcal{G}_1$ of all transformations of the space T_0 preserving the double ratio of four vectors, and the group $\mathcal{G}_2$ of all transformations of the space T_0 preserving the measure of k-angles, have a precise meaning.

Theorem 5. *If the Lie algebras G_1 and G_2 coincide, then the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ also coincide (as a whole).*

For those cases when $N(x)$ is the general norm of a semisimple Jordan algebra S, one can determine the concrete structure of the transformations belonging to the groups $\mathcal{G}_1$ and $\mathcal{G}_2$.

Theorem 6. *Every transformation in the space of a semisimple Jordan algebra S, defined in a neighborhood of zero and preserving the double ratio (1), constructed for the general norm $N(x)$ of the algebra S, is uniquely represented in the form*

$$F(x) = A((a - x^{-1})^{-1}) + b,$$

where a, b are certain elements of the algebra S, the inverse element being taken in the sense of the algebra S, and $A(x)$ is a linear transformation satisfying the condition

$$N(A(x), A(x), \dots, A(x)) = \lambda N(x, x, \dots, x), \tag{6}$$

where λ is a number. Conversely, for arbitrary $a, b, A(x)$ the transformation (6) preserves the double ratio of four vectors.

These same transformations, and only they, preserve the measure of k-angles if the algebra S contains no one-dimensional ideals in the complex case and contains no one-dimensional or two-dimensional ideals in the real case.

We shall give several theorems that follow from Theorems 4 and 6 when considering the general norms of simple Jordan algebras and the corresponding transitively differential groups.

Consider the exceptional Jordan algebra defined with respect to the operation $AB + BA$ on the space of Hermitian octavic matrices of the third order of the form

$$\begin{pmatrix} \alpha_1 & a_3 & \bar{a}_2 \\ \bar{a}_3 & \alpha_2 & a_1 \\ a_2 & \bar{a}_1 & \alpha_3 \end{pmatrix}, \tag{7}$$

where the α_i are numbers, the a_i are octaves, and $\bar{a}_i$ denotes the octave conjugate to a_i. The general norm of this algebra is the cubic form

$$N(A) \equiv N(A, A, A) = \alpha_1\alpha_2\alpha_3 + T((a_1a_2)a_3) - \alpha_1\tilde{N}(a_1) -$$

$$-\alpha_2\tilde{N}(a_2) - \alpha_3\tilde{N}(a_3), \tag{8}$$

where $T(a)$ and $\tilde{N}(a)$ are the trace and norm of the octave a.

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Theorem 7. *The group of all transformations of the space of Hermitian octavic matrices of the third order preserving the measure of 3-angles,*

$$\frac{N(A,B,C)^3}{N(A)N(B)N(C)},$$

where $N(A)$ is the form (8), coincides with the group of all transformations of this

spaces preserving the double ratio

$$\frac{N(X-A,\,X-A,\,X-A)}{N(Y-A,\,Y-A,\,Y-A)} : \frac{N(X-B,\,X-B,\,X-B)}{N(Y-B,\,Y-B,\,Y-B)}$$

and is the complex group E_7, when the a_i are complex numbers, the a_i are complex octaves, and the noncompact real form of the group E_7, of types respectively E_7^3 and E_7^1, when the a_i are real numbers, and the a_i are real octaves or pseudo-octaves.

Theorem 8. The group of all conformal transformations of an n-dimensional linear space with a given nondegenerate quadratic form (x,x) coincides with the group of all transformations of this space preserving the double ratio of four points

$$\frac{(x-a,\,x-a)}{(y-a,\,y-a)} : \frac{(x-b,\,x-b)}{(y-b,\,y-b)}.$$

In the following theorems, by the determinant of a quaternion matrix is meant the determinant of the complex matrix obtained from the given one by replacing each quaternion $a+bi+cj+dk$ by the matrix

$$\begin{pmatrix} a+bi & c-di \\ -c-di & a-bi \end{pmatrix}.$$

Theorem 9. The group of all transformations of the space of complex, real, or quaternion matrices of order n preserving the double ratio

$$\frac{|X-A|}{|Y-A|} : \frac{|X-B|}{|Y-B|}, \tag{9}$$

where $|X|$ is the determinant of the matrix X, coincides with the group of all transformations preserving the measure of n-angles

$$\frac{\varphi(A_1,A_2,\dots,A_n)^n}{|A_1|\,|A_2|\cdots|A_n|}, \tag{10}$$

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where $\varphi(A_1, A_2, \dots, A_n)$ is the n-linear form obtained by polarizing the determinant (in the quaternion case, the $2n$-linear form). All transformations have the form

$$X' = (AX + B)(CX + D)^{-1}, \qquad X' = (AX^T + B)(CX^T + D)^{-1},$$

where A, B, C, D are matrices of order n, respectively complex, real, or quaternionic, such that

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \pm 1.$$

Theorem 10. The group of all transformations of the space of real skew-symmetric, complex skew-symmetric matrices of order $n = 2k$, preserving the double ratio (9), coincides with the group of all transformations preserving the measure of n-angles (10). The transformations of the group have the form $X' = (AX + B)(CX + D)^{-1}$, where the matrices A, B, C, D of order n are connected by the conditions

$$A^T C + C^T A = 0, \qquad B^T D + D^T B = 0, \qquad A^T D + C^T B = \pm E.$$

Theorem 11. The group of all transformations of the space of real symmetric, complex Hermitian, quaternionic Hermitian matrices of order n, preserving the double ratio (9), coincides with the group of all transformations preserving the measure of n-angles (10) and has the form $X' = (AX + B)(CX + D)^{-1}$, where A, B, C, D are matrices of order n, respectively real, complex, and quaternionic, satisfying the conditions

$$A^* C - C^* A = 0, \qquad B^* D - D^* B = 0, \qquad A^* D - C^* B = \pm E,$$

where A^* denotes the matrix A transposed and conjugated, respectively complex or quaternionic.

All-Union Correspondence
Financial-Economic Institute

Received
28 III 1966

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