# ON THE EMBEDDING OF RINGS IN FIELDS

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## Abstract

## Full Text

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*MATHEMATICS*

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# ON THE EMBEDDING OF RINGS IN FIELDS

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I. In this note a proof is given of the following result:

**Theorem.** *There exists an (associative) ring without zero divisors, not embeddable in a field, whose multiplicative semigroup of nonzero elements is embeddable in a group.*

This is a solution of a problem posed by A. I. Mal' tsev.

II. Following (2), let us call a ring $K$ an $SN$-ring if $K$ is an algebra with 1 over a field $F$, given in some system of generators $\Xi$ by relations of the form

$$w_i h_i = u_i f_i, \tag{1}$$

where $i \in I,\ w_i, h_i, u_i, f_i \in \Xi$, satisfying the conditions:

S1) $\{w_i, u_i\} \cap \{h_j, f_j\} = \emptyset$.

S2) The words $w_i h_i,\ u_i f_i,\ i \in I$, are pairwise distinct.

N) If in $K$ there is an equality $AB = CD$, where $A, B, C, D$ are nonzero linear combinations of generators from $\Xi$, then either $A = CX,\ D = XB,\ X \in K$, or there is an equality $xy = zt$ among (1) such that the relations hold:

$$A = aX(x + \alpha z), \quad B = (y + \beta t)Y, \quad C = X(z + \beta x), \quad D = (t + \alpha y)Ya, \tag{2}$$

where $0 \neq X, Y, a \in F,\ \alpha, \beta \in F$.

In (1) it is proved that, as a basis of the algebra $K$, one may take the element 1 and all words in the generators not containing the left-hand sides of the relations (1) (assuming that in (1) the left- and right-hand sides are fixed). In (2) it is proved that $SN$-rings have no zero divisors.

By $\overline{SN}$ we shall denote the totality of completions $\overline{K}$ of rings $K \in SN$ with respect to the natural norm. The ring $\overline{K}$ is the ring of formal infinite series of the form

$$A = \sum_{i=0}^{\infty} A_i,$$

where $A_i$ are homogeneous elements of degree $i$.

III. Let $\overline{K}$ be an arbitrary $\overline{SN}$-ring. Introduce the following sets of elements of the ring $\overline{K}$: $\Gamma$ is the set of all invertible elements of the ring $\overline{K}$; $P$ is the set of elements of the form

$$p(w, u, A) = w + uA, \quad p(u, w, C) = u + wC, \tag{3}$$

where $wh = uf$ is an arbitrary one of the relations (1); $A$ is an arbitrary element of $\overline{K}$ such that all basic words in $A$ do not begin with $f$ and do not end with $w$; all basic words in $C$ do not begin with $h$ and do not end with $w$; $Q$ is the set of elements of the form

$$q(f, h, A) = f + Ah, \quad q(h, f, C) = h + Cf, \tag{4}$$

where $wh = uf$ is an arbitrary one of the relations (1), and $A$ and $C$ are the same as above; $R$ is the set of all irreducible elements (one from each class of associated elements), each of which is not associated with any element from $P \cup Q$.

Let us single out in the sets $P, Q$ the subsets $P \subseteq P_1, \ P^*, \ Q \supseteq Q_1, \ Q^*$. Namely, into $P_1$ and $Q_1$ we place elements of the form (3) and (4) for which $A$ and $C$ are not invertible; into $P^*, Q^*$ we place elements of the form $P^* = \{p(w, u, A^*)\}, \ Q^* =$

$$= \{q(f, h, A^*)\},$$

where $A^*$ are invertible elements. For a fixed relation

$$wh = uf \tag{5}$$

we shall regard the sets of all noninvertible elements $A$ and $C$ occurring in (3) and (4), as well as the set of all invertible elements $A^*$ from (3) and (4), as completely ordered:

$$A_1, A_2, \ldots; \ C_1, C_2, \ldots; \ A_1^*, A_2^*, \ldots.$$

After this we introduce new notation for the elements of the sets $P_1, Q_1, P^*, Q^*$, whose meaning will be clear from the examples:

$$p(w, u, A_i) = p_i(w, u) = p_{iw}, \qquad q(f, h, A_i^*) = q_i^*(f, h) = q_{if}^*.$$

We define the group $\Gamma$ by a system of generators consisting of all elements of this group and by some system of relations $X_i = 1$. Denote by $\overline{K}^*$ the multiplicative semigroup of nonzero elements of the ring $\overline{K}$.

**Theorem 1** ([3]). *The semigroup $\overline{K}^*$ is given in the system of generators*

$$P_1 \cup Q_1 \cup P^* \cup Q^* \cup R \cup \Gamma \tag{6}$$

*by the following system of defining relations, constructed for all relations (5) among (1) and for all indices $k, i, n, m$:*

$$1. \quad p_k(w,u) X_{ki} q_i(h,f) = p_i(u,w) Y_{ik} q_k(f,h). \tag{7}$$

$$2. \quad p_k^*(w,u) X_{ni}^* q_i(h,f) = p_i(u,w) Y_{in}^* q_n^*(f,h). \tag{8}$$

$$3. \quad p_k(w,u) \overline{X}_{kn} q_n^*(f,h) = p_n^*(w,u) \overline{Y}_{nk} q_k(f,h). \tag{9}$$

$$4. \quad p_n^*(w,u) X_{nm}^{**} q_m^*(f,h) = p_m^*(w,u) Y_{mn}^{**} q_n^*(f,h), \tag{10}$$

*where $A_n^* - A_m^*$ is an invertible element,*

$$X_{nm}^{**} = Y_{mn}^{**} = (A_n^* - A_m^*).$$

$$5. \quad \alpha(a + L\alpha) = (a + \alpha L)\alpha, \tag{11}$$

*where* $0 \neq a \in F,\ \alpha \in P_1 \cup P^* \cup Q_1 \cup Q^* \cup R$.

$$6. \quad X_i = 1. \tag{12}$$

Here we have used the notation:

$$X_{ki} = (1 - C_{iA}k)^{-1}, \qquad Y_{ik} = (1 - A_{kC}i)^{-1}, \qquad X_{ni}^* = (1 - C_{iA}n^*)^{-1}, \qquad Y_{in}^* = (1 - A_n^* C_i)^{-1},$$

$$\overline{X}_{nk} = \overline{Y}_{kn} = (A_n^* - A_k)^{-1}.$$

IV. We now recall the definition of the rings $F(Q_n)$ from ([1]). The ring $F(Q_n)$, $n \geq 2$, is an algebra with 1 over the field $F$, given in the system of generators

$$\{a_i, b_i, c_i, s_i, t_i,\ 1 \leq i \leq n;\ v_0, v_1\}$$

by the defining relations

$$a_{is}i = c_{iv}0, \qquad b_{is_{i+1}} = c_{iv}1, \qquad b_{it_{i+1}} = a_{it}i,$$

where $1 \leq i \leq n,\ s_{n+1} = s_1,\ t_{n+1} = t_1$. In ([2]) it was proved that the rings $F(Q_n)$ are $SN$-rings. If $F$ is a field of characteristic $p \neq 0,\ n = p^k$, then, as shown in ([1]), the rings $F(Q_n)$ are not embeddable in fields. On the other hand, it will follow from what follows that the semigroups $GF(2)(Q_n)^*$ are embeddable in groups. Therefore the ring $GF(2)(Q_4)$ is the simplest example of a ring giving an answer to A. I. Mal' cev' s problem.

V. Let again $\overline{K}$ be an arbitrary $\overline{SN}$-ring, and let $G$ be the group of fractions of the semigroup $\overline{K}^*$, i.e. the group given by the generators (6) and the relations (7)–(12). Fix some relation (5). In the group $G$ the equalities

$$\begin{aligned} &X^*_{n1} q_{1n} Q^*_{1nf} q^{-1}_{1h} X^{*-1}_{ni} P'_{1nw} \overline{Y}_{nk} q_{kf} Q^*_{2nf} q^{-1}_{sf} \overline{Y}^{*-1}_{ns} P'_{2nw}, \\ &P'_{3nw} \overline{Y}_{nk} q_{kf} Q^*_{3nf} q^{-1}_{sf} \overline{Y}^{*-1}_{ns} P'_{4nw} X^*_{n1} q_{1h} Q^*_{4ng} q^{-1}_{ih} X^{*-1}_{ni}, \end{aligned} \tag{13}$$

hold, if the following relations hold:

$$\begin{aligned} &1.\quad P^*_{1nw} = P_{iu} P_{kw}. \\ &2.\quad Y^*_{ln} Q^*_{1nf} Y^{*-1}_{in} P'_{iu} = P'_{lu} Y_{lk} Q_{kf} Y^{-1}_{ik}. \\ &3.\quad P'_{kw} \overline{X}_{kn} Q^*_{2nt} \overline{X}^{*-1}_{sn} = X_{ki} Q_{in} X^{-1}_{si} P'_{sw}. \\ &4.\quad Q'_{kf} Q'_{in} = Q'_{ln} Q'_{sf}. \end{aligned} \tag{14}$$

5. $P_{1u} = P^*_{3nw} P_{2kw}$.
6. $P_{1iu} = P^*_{sw} P^*_{2nw}$.
7. $P'_{1kw} X_{kl} Q_{ln} X^{-1}_{sl} = \overline{X}^*_{kn} Q^*_{3nf} \overline{X}^{*-1}_{sn} P'_{1sw}$.
8. $Y_{ls} Q_{sf} Y^{-1}_{is} P'_{1iu} = P'_{1lu} Y^*_{ln} Q^*_{4nf} Y^{*-1}_{in}$.
9. $P_{1sw} P_{1lu} = P^*_{4nw}$.

The notation introduced here will be clear from the following examples: $Q^*_{1nf} = 1 + q^*_{nf} L_1$, $Q^{*'}_{1nf} = 1 + L_1 q^*_{nf}$, $Q^*_{2nf} = 1 + q^*_{nf} L_2$, where $L_1, L_2 \in \overline{K}$, $P_{iu} = 1 + p_{iu} L$, $P'_{iu} = 1 + L p_{iu}$, $L \in \overline{K}$.

In $G$ the equalities

$$\begin{aligned} &\overline{X}^{*-1}_{sn} p^{-1}_{sw} \overline{P}^*_{1nw} p_{kw} \overline{X}^*_{kn} Q^*_{1nf} Y^*_{in} p^{-1}_{iu} P^*_{2nw} p_{lu} Y^*_{ln} Q_{2nf} = \\ &= Q^*_{3nf} Y^{*-1}_{in} p^{-1}_{iu} P^*_{3nw} p_{lu} Y^*_{ln} Q^*_{4nf} \overline{X}^{*-1}_{sn} p^{-1}_{sw} P^*_{4nw} p_{kw} \overline{X}^*_{kn}, \end{aligned} \tag{15}$$

also hold, if relations analogous to those given in (14) hold.

VI. Let us give one more collection of equalities of the group $G$. Suppose that among the defining relations of the ring $K$ under consideration there is a triple of equalities of the form:

$$wh = uf, \qquad w_1h_1 = uf_1, \qquad wh_2 = w_1f_2. \tag{16}$$

Fix the equalities (16) and introduce the notation:

$$\begin{array}{lll} \overline{p}_{nw} = w + uf_1L_n, & \overline{q}_{nf} = f + f_1L_nh, & \overline{q}_{nf_2} = f_2 + h_1L_nh_2, \\ \overline{p}_{iu} = u + wh_2M_i, & \overline{q}_{ih} = h + h_2M_if, & \overline{q}_{ih_1} = h_1 + f_2M_if_1, \\ \overline{p}_{kw_1} = w_1 + ufS_k, & \overline{q}_{kf_1} = f_1 + fS_kh, & \overline{q}_{kh_2} = h_2 + h_1S_kf_2, \end{array} \tag{17}$$

where $L_n, M_i, S_k$ are arbitrary nonzero elements of the ring $\overline{K}$, not ending in $w, u, w_1$, respectively. (We regard the sets $\{L_n\}$, $\{M_i\}$, $\{S_k\}$ as completely ordered.)

In the group $G$ the equalities

$$P'_{1nw}X'_{nj}Q'^{-1}_{\overline{nf}}\overline{q}^{-1}_{in}X'^{-1}_{ni}P'_{nw}X'''_{nl}q_{lh_2}Q'^{-1}_{\overline{nf_2}}\overline{q}^{-1}_{kh_2}X'''^{-1}_{nk} =$$

$$= X'''_{nl}q_{lh_2}Q'^{-1}_{\overline{nf_2}}\overline{q}^{-1}_{kh_2}X'''^{-1}_{nk}P'_{3nw}X'_{ni}\overline{q}_{jh}Q'^{-1}_{\overline{nf}}\overline{q}^{-1}_{ih}X'^{-1}_{ni}P'_{2hw}, \tag{18}$$

hold, if relations analogous to those given in (14) are satisfied.

The meaning of the notation introduced will be clear from examples:

$$P_{1nw} = 1 + \overline{p}_{nw}A_1, \quad P'_{1nw} = 1 + A_1\overline{p}_{nw}, \quad A_1 \in \overline{K}, \quad X'_{ni} = (1 - h_2M_ifL_n)^{-1},$$

$$Y'_{in} = (1 - f_1L_nh_2M_i)^{-1}, \quad X''_{ki} = (1 - f_2M_ifS_k)^{-1}, \quad X'''_{nk} = (1 - hS_kh_1L_n)^{-1}.$$

VII. Let now $F = GF(2)$. Fix some semigroup $\overline{F(Q_n)}^*$, $n \geq 2$, and let $G$ be its group of quotients. We shall show how, on the basis of what has preceded, it is proved that the semigroup $\overline{F(Q_n)}^*$ is a subsemigroup of the group $G$. First of all, include among the defining relations of the group $G$ the relations (13), (15), and (18). Next define a finite sequence of groups $G_i$, $1 \leq i \leq 8$. All these groups are obtained according to the rule: among the generators of the group $G$ some subset is singled out and all defining relations from the elements of this subset are considered. Therefore, in order to define the groups $G_i$, it is enough to specify the sets of generators $\Sigma_i$ of each of these groups. We have: $\Sigma_1 = \Xi \cup \Gamma$; $\Sigma_2$ is obtained from $\Sigma_1$ by adjoining all elements of the form $p_i(u, w)$, $q_i(h, f)$ such that they do not enter the sets (18), constructed for all relations (17) obtained from the relations of the ring $F(Q_n)$ by fixing the number $i$, $1 \leq i \leq n$; $\Sigma_3$ is obtained by adjoining to $\Sigma_2$ elements of the form $\overline{p}_n(u, w)$, $\overline{q}_n(h, f)$, $\overline{q}_n(h_1, f_1)$, constructed for all relations (17);

$\Sigma_4$ is obtained from $\Sigma_3$ by adding all elements $p_n(w,u)$, $q_n(f,h)$ that do not belong to the set (18); $\Sigma_5$ consists of the elements of $\Sigma_4$ and elements of the form $\bar{p}_n(w_1,u)$, $\bar{q}_n(f_1,h_1)$, $\bar{q}_n(h_2,f_2)$, constructed for all triples (17); $\Sigma_6$ is obtained by adding to $\Sigma_5$ the elements $\bar{p}_n(w,u)$, $\bar{q}_n(f,h)$, $\bar{q}_n(f_2,h_2)$, constructed for all triples (17); finally,

$$\Sigma_7 = \Sigma_6 \cup P^* \cup Q^*, \quad \Sigma_8 = \Sigma_7 \cup R.$$

In the groups $G_i$, $1 \le i \le 8$, a canonical form of elements is constructed successively, analogously to how this is done in the works ([1]) and, especially, ([4]). As a result it turns out that every word of the group $G$ is equal to one and only one word having canonical form. Moreover, if $W$ is a positive word of the group $G$ (a word in the alphabet (6)), then to obtain its canonical form it is enough to carry out on $W$ a certain number of "semigroup" transformations (transformations corresponding to the equalities (7)–(12)). This proves the required result.

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*