ON RELATIONS ON STRESS-DISCONTINUITY SURFACES IN THREE-DIMENSIONAL IDEAL RIGID-PLASTIC BODIES

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Abstract

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ON RELATIONS ON STRESS-DISCONTINUITY SURFACES IN THREE-DIMENSIONAL IDEAL RIGID-PLASTIC BODIES

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Discontinuous solutions of the equations of the theory of ideal plasticity have repeatedly been used in solving many problems of plane strain, plane stress, and torsion of prismatic bars. Examples of the use of discontinuous solutions in these cases are well known ([1–3]). In ([4]), relations on a stress-discontinuity surface for three-dimensional bodies were studied, for stress states corresponding to an edge of the Tresca prism. In ([5]) it was shown that on a stress-discontinuity surface, for convex plasticity conditions, the displacements are continuous and the plastic strain rates are equal to zero. Below, relations on a stress-discontinuity surface are derived for an arbitrary plasticity condition, and consequences of these relations are obtained for the Mises and Tresca plasticity conditions.

1. Let there be, in a three-dimensional plastic body, a surface Σ on which the stresses undergo a discontinuity. On the discontinuity surface the contact stresses must be continuous, whence it follows that

$$[\sigma_{ij}]v_j = 0, \tag{1,1}$$

where $[\sigma_{ij}] = \sigma_{ij}^- - \sigma_{ij}^+$ is the difference of the stresses on the different sides of the surface Σ; v_j is the unit normal to this surface. The stresses on both sides of the surface must satisfy the plasticity condition

$$f(\sigma_{ij}) = 1, \tag{1,2}$$

whence it follows that

$$[f(\sigma_{ij})] = f(\sigma_{ij}^-) - f(\sigma_{ij}^+) = 0. \tag{1,3}$$

The strain rates in a rigid-plastic body are related to the stresses by the associated flow law

$$\varepsilon_{ij} = {}^1\!/_2(u_{i,j} + u_{j,i}) = \lambda p_{ij}, \quad (1{,}4)$$

where $p_{ij} = \partial f/\partial \sigma_{ij}$; u_i are the projections of the displacement velocity; the comma denotes differentiation with respect to a coordinate; λ is an undetermined multiplier greater than zero.

Relations (1,4) hold on the different sides of the surface Σ, therefore

$$[\varepsilon_{ij}] = {}^1\!/_2([u_{i,j}] + [u_{j,i}]) = [\lambda p_{ij}]. \quad (1{,}5)$$

Using the geometrical compatibility conditions ([6]) for the derivatives of the displacement velocities, $[u_{i,j}]$ may be represented in the form

$$[u_{i,j}] = \lambda_i v_j, \quad \text{where} \quad \lambda_i = [u_{i,j}] v_j. \quad (1{,}6)$$

From relations (1,5) and (1,6) it follows that

$$[\varepsilon_{ij}] = {}^1\!/_2(\lambda_i v_j + \lambda_j v_i) = [\lambda p_{ij}]. \quad (1{,}7)$$

If the yield surface is convex, then by (5) $\varepsilon_{ij}^+ = \varepsilon_{ij}^- = 0$, whence it follows that $\lambda^+ = \lambda^- = 0$. Equating the indices i and j in relations (1.6), after summation we obtain $\lambda_i v_i = 0$. Multiplying equality (1.7) by v_j and summing with respect to j, we obtain $\lambda_i = 0$; from conditions (1.6) it then follows that the first derivatives of the displacement velocities are continuous on the surface Σ.

In what follows we shall restrict ourselves to the consideration of isotropic materials; in this case the stress tensor and the strain-rate tensor will be coaxial.

Equations (1.1), (1.3), (1.7), generally speaking, are insufficient for determining the stress state behind the discontinuity surface Σ, since the tensor ε_{ij} may vanish on this surface.

Let, on the stress discontinuity surface Σ, the strain rates ε_{ij}^+ and ε_{ij}^- vanish. Since plastic zones are assumed on both sides of Σ, there must exist derivatives of the strain rates on the surface Σ different from zero. Let $\varepsilon_{ij,k\dots l} \neq 0$; then, differentiating relations (1.4), we obtain

$$\varepsilon_{ij,k\dots l} = \frac{1}{2}(u_{i,jk\dots l} + u_{j,ik\dots l}) = \lambda_{,k\dots l} p_{ij}. \quad (1.8)$$

Writing (1.8) in discontinuities and using the geometrical compatibility conditions, it is easy to obtain the relations

$$[\varepsilon_{ij,k\dots l}] = \frac{1}{2}(c_i v_j v_k \dots v_l + c_j v_i v_k \dots v_l) = [\lambda_{,k\dots l} p_{ij}];$$

$$c_i[u_{i,jk\dots l}]v_{jv}k \dots v_l. \tag{1.9}$$

Multiplying relations (1.9) by $v_k \dots v_l$ and summing over repeated indices, we obtain the equalities

$$[a_{ij}] = \frac{1}{2}(c_i v_j + c_{jv} i) = [\psi p_{ij}];$$

$$a_{ij} = \varepsilon_{ij,k\dots l} v_k \dots v_l; \qquad \psi = \lambda_{,k\dots l} v_k \dots v_l. \tag{1.10}$$

Equating i and j in relations (1.10), using the incompressibility condition, after summation we obtain

$$[a_{ii}] = c_i v_i = [\psi p_{ii}] = 0. \tag{1.11}$$

From the system (1.10) and (1.11) we find the quantities c_i

$$c_i = 2[\psi p_{ij}]v_j;$$

then relation (1.10) may be represented in the form

$$[\psi p_{ik}]v_{kv}j + [\psi p_{jk}]v_{kv}i = [\psi p_{ij}]. \tag{1.12}$$

Among these relations only 3 are linearly independent; therefore equations (1.1), (1.3), (1.12) determine a closed system for finding the 7 unknown quantities: σ_{ij}^-, ψ^-.

In the canonical coordinate system ($v_1 = v_2 = 0;\ v_3 = 1$) the system of equations (1.1), (1.3), (1.12) takes the form

$$[\sigma_{i3}] = 0; \qquad [f(\sigma_{ij})] = 0; \qquad [\psi p_{11}] = [\psi p_{22}] = [\psi p_{12}] = 0. \tag{1.13}$$

2. As an example, consider stress discontinuities in a plastic body under the Mises plasticity condition. Relations (1.13) in this case take the form

$$[\sigma_{i3}] = 0; \qquad [s_{ij}s_{ij} - 2k^2] = 0; \qquad [\psi s_{11}] = [\psi s_{22}] = [\psi s_{12}] = 0, \tag{2.1}$$

where $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma\delta_{ij}$; δ_{ij} is the Kronecker symbol.

A combination of relations (2.1) gives the equality

$$\{1 - (\psi^+/\psi^-)^2\}(s_{11}^2 + s_{22}^2 + s_{33}^2 + 2s_{12}^2) = 0. \tag{2.2}$$

From relations (2.2) and (2.1) it follows that a stress discontinuity is possible only when $\psi^+/\psi^- = -1$, and the corresponding expressions for the stresses have the form:

$$\begin{gathered}\sigma_{11}^- = 2\sigma_{33}^+ - \sigma_{11}^+; \qquad \sigma_{22}^- = 2\sigma_{33}^+ - \sigma_{22}^+; \qquad \sigma_{33}^- = \sigma_{33}^+; \\ \sigma_{12}^- = -\sigma_{12}^+; \qquad \sigma_{13}^- = \sigma_{13}^+; \qquad \sigma_{23}^- = \sigma_{23}^+.\end{gathered} \tag{2.3}$$

If the direction cosines of the principal axes $l_i;\ m_i;\ n_i$ are introduced, then the stresses σ_{ij} can be represented as follows:

$$\sigma_{ij} = \sigma_1 l_i l_j + \sigma_2 m_i m_j + \sigma_3 n_i n_j, \tag{2.4}$$

where $\sigma_1,\ \sigma_2,\ \sigma_3$ are the principal stresses. Substituting relations (2.4) into equations (2.3) and taking into account that

$$l_i l_j + m_i m_j + n_i n_j = \delta_{ij}, \tag{2.5}$$

we obtain a system of 12 equations with respect to $\sigma_1^-;\ \sigma_2^-;\ \sigma_3^-;\ l_i^-;\ m_i^-;\ n_i^-$. The solution of this system is represented in the form

$$\begin{gathered}\sigma_1^- = 2\sigma_{33} - \sigma_1^+; \qquad \sigma_2^- = 2\sigma_{33} - \sigma_2^+; \qquad \sigma_3^- = 2\sigma_{33} - \sigma_3^+; \\ l_1^- = \pm l_1^+; \qquad m_1^- = \pm m_1^+; \qquad n_1^- = \mp n_1^+; \\ l_2^- = \pm l_2^+; \qquad m_2^- = \pm m_2^+; \qquad n_2^- = \mp n_2^+; \\ l_3^- = \mp l_3^+; \qquad m_3^- = \mp m_3^+; \qquad n_3^- = \pm n_3^+,\end{gathered} \tag{2.6}$$

i.e., the corresponding principal axes are equally inclined to the discontinuity surface Σ and lie in planes passing through the normal to Σ. It is necessary to note that relations (2.6), which relate the stress states on different sides of Σ, correspond to points lying on diametrically opposite sides of the yield surface.

3. Under a plasticity condition that includes plane portions and edges, the strain rates on the stress-discontinuity surface may be nonzero and may undergo a discontinuity. However, the defining relations (1.12) completely retain their form.

Let the stress state on different sides of the surface Σ satisfy an edge of the Tresca plasticity condition. Without loss of generality, suppose that the maximum value of the difference is attained between the second and third principal stresses; then

$$\sigma_2 - \sigma_3 = \pm 2k. \tag{3.1}$$

Using relations (2.4), (3.1), the system (1.13) can be represented in the form

$$
\begin{aligned}
[\sigma_{33}] &= [\sigma_1 l_3^2 + \sigma_2 m_3^2 + \sigma_3 n_3^2] = 0;\\
[\sigma_{13}] &= [\sigma_1 l_1 l_3 + \sigma_2 m_1 m_3 + \sigma_3 n_1 n_3] = 0;\\
[\sigma_{23}] &= [\sigma_1 l_2 l_3 + \sigma_2 m_2 m_3 + \sigma_3 n_2 n_3] = 0;
\end{aligned} \tag{3.2}
$$

$$
[\varepsilon_{11}] = [\lambda(m_1^2 - n_1^2)] = 0; \qquad [\varepsilon_{22}] = [\lambda(m_2^2 - n_2^2)] = 0; \tag{3.3}
$$

$$
[\varepsilon_{12}] = [\lambda(m_1 m_2 - n_1 n_2)] = 0.
$$

Relations (2.5), (3.1), (3.2), (3.3) give a closed system of equations for determining the stress and strain state beyond the discontinuity surface. We note that they must be invariant with respect to rotation of the coordinate system about the third axis. Rotating the coordinate system so that σ_{13} becomes zero, and combining (3.1) with the second and third equations of (3.2), we obtain:

$$
(m_3^+ n_3^+ / l_1^+)^2 = (m_3^- n_3^- / l_1^-)^2. \tag{3.4}
$$

Relations (3.3), after eliminating λ from them, split into two systems of linear equations

$$
\begin{aligned}
(m_1^+ - n_1^+)/(m_2^+ - n_2^+) &= (m_1^- - n_1^-)/(m_2^- - n_2^-);\\
(m_1^+ + n_1^+)/(m_2^+ + n_2^+) &= (m_1^- + n_1^-)/(m_2^- + n_2^-);
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
(m_1^+ - n_1^+)/(m_2^+ - n_2^+) &= (m_1^- + n_1^-)/(m_2^- + n_2^-);\\
(m_1^+ + n_1^+)/(m_2^+ + n_2^+) &= (m_1^- - n_1^-)/(m_2^- - n_2^-).
\end{aligned} \tag{3.6}
$$

The system of equations (2.4), (3.4), (3.5) is satisfied by the solution

$$
l_i^- = \pm l_i^+; \qquad m_i^- = \pm m_i^+; \qquad n_i^- = \pm n_i^+, \tag{3.7}
$$

and the equations (2.4), (3.4), (3.6) are satisfied by the solution (2.6). Substituting (3.7) into relations (3.2), we obtain

$$
\begin{aligned}
[\sigma_1]\, l_3^2 + [\sigma_2]\, m_3^2 + [\sigma_3]\, n_3^2 &= 0;\\
[\sigma_1]\, l_1 l_3 + [\sigma_2]\, m_1 m_3 + [\sigma_3]\, n_1 n_3 &= 0;\\
[\sigma_1]\, l_2 l_3 + [\sigma_2]\, m_2 m_3 + [\sigma_3]\, n_2 n_3 &= 0.
\end{aligned} \tag{3.8}
$$

Fig. 1

Since a stress discontinuity on the surface Σ is assumed, the determinant of system (3.8) must be equal to zero, whence

$$l_3 m_3 n_3 = 0, \quad (3.9)$$

i.e., one or two of the principal axes lie in the plane tangent to Σ.

Solving jointly the system of equations (3.8), (3.9), (3.1), for stress discontinuities we obtain different solutions:

$$l_3 = 0; \quad m_3 \neq 0; \quad n_3 \neq 0; \quad [\sigma_2] = [\sigma_3] = 0; \quad [\sigma_1] \neq 0; \quad (3.10)$$

$$l_3 \neq 0; \quad m_3 = 0; \quad n_3 \neq 0; \quad [\sigma_1] = [\sigma_3] = 0; \quad [\sigma_2] = \pm 4k; \quad (3.11)$$

$$l_3 = m_3 = 0; \quad n_3 = 1; \quad [\sigma_3] = [\sigma_2] = 0; \quad [\sigma_1] \neq 0; \quad (3.12)$$

$$l_3 = m_3 = 0; \quad n_3 = 1; \quad [\sigma_3] = 0; \quad [\sigma_2] = \pm 4k; \quad [\sigma_1] \neq 0; \quad (3.13)$$

$$m_3 = n_3 = 0; \quad l_3 = 1; \quad [\sigma_1] = 0; \quad [\sigma_2] = [\sigma_3] \neq 0; \quad (3.14)$$

$$m_3 = n_3 = 0; \quad l_3 = 1; \quad [\sigma_1] = 0; \quad [\sigma_2] = [\sigma_3] \pm 4k, \quad (3.15)$$

where the solutions (3.11), (3.13), (3.15) correspond to different signs before $2k$ in relations (3.1) on opposite sides of the stress-discontinuity surface Σ, while the solutions (3.12), (3.14) correspond to the same sign before $2k$.

From relations (3.1), (2.6) it follows that, on the stress-discontinuity surface, a discontinuity of the strain rates ε_{13} and ε_{23} is possible,

$$[\varepsilon_{13}] = -2\varepsilon_{13}^{+}; \quad [\varepsilon_{23}] = -2\varepsilon_{23}^{+}.$$

The change of $[\varepsilon_{13}]$ and $[\varepsilon_{23}]$ in the plane tangent to Σ is shown in Fig. 1.

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Note: Figure translations are in progress. See original paper for figures.