LINEAR DIFFERENTIAL GAMES

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Abstract

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LINEAR DIFFERENTIAL GAMES

1. Let the motion of the vector z in an n-dimensional Euclidean space R be described by the linear vector differential equation

$$dz/dt = Cz + U(u) - V(v), \tag{1}$$

where u and v are control parameters whose values lie on the $(\nu-1)$-dimensional unit sphere K; C is a constant matrix; U and V are analytic mappings of the sphere K into the space R. Suppose that in R an $(n-\nu)$-dimensional vector subspace M is given. We shall say that all these data describe a **linear differential game** (1) (see (1)).

We shall say that the game (1) can be completed on a certain set $A \subset R$ if, for any initial value $z_0 \in A$ of the vector z and for an arbitrary piecewise-continuous variation of the control parameter $v(t)$, one can choose such a variation of the control parameter $u(t)$ that the point z reaches the subspace M in a time not exceeding the number $T(z_0) \geq 0$; moreover, in order to find the value of the parameter $u(t)$ at each time instant t, only the values $z(t)$ and $v(t)$ at the same time instant t are used, and the values of z and v at times following t are not used.

In the present note we indicate some very simple conditions sufficient for the completion of a linear differential game.

2. Let L be the orthogonal complement in R to the subspace M. We shall denote by πz the orthogonal projection of any vector $z \in R$ onto the subspace L. The right-hand side of equation (1) determines the following two mappings of the sphere K into L:

$$y = \pi\left(e^{-\tau C} U(u)\right), \tag{2}$$

$$y = \pi\left(e^{-\tau C} V(v)\right). \tag{3}$$

Everywhere below we shall assume that τ is a negative number.

We shall suppose that the following condition is fulfilled.

Condition a). The images of the sphere K under the mappings (2) and (3) are convex, locally convex $(\nu-1)$-dimensional hypersurfaces in L. (Local convexity is understood here in the differential-geometric sense, as the definiteness of the second quadratic form.)

Consider the scalar products

$$\varphi \cdot \pi\left(e^{-\tau C} U(u)\right), \qquad \varphi \cdot \pi\left(e^{-\tau C} V(v)\right), \tag{4}$$

where φ is an arbitrary unit vector from L, which we shall regard as a point of the sphere K, and the sphere K as the unit sphere in L. Let $u = u(\tau, \varphi)$, $v = v(\tau, \varphi)$ be the points giving the maximum of the corresponding expressions (4) for fixed τ, φ, with respect to u, respectively to v. Construct the mapping

$$y = \pi\left[e^{-\tau C}\left(U(u(\tau, \varphi)) - V(v(\tau, \varphi))\right)\right] \equiv w(\tau, \varphi) \tag{5}$$

of the sphere K into L. This mapping, generally speaking, is neither regular nor one-to-one. Denote by Σ_τ the image of the sphere K under the mapping—

of mapping (5). It is easy to see that the vector φ is the normal to the surface Σ_τ at the point $w(\tau, \varphi)$. We shall assume that the following is satisfied.

Condition b). The surface Σ_τ is locally convex, and the vector φ is the outward normal to Σ_τ at the corresponding point.

Finally, we shall assume that one more condition is satisfied.

Condition c). The orthogonal projection of the subspace CM onto L coincides with all of L.

Then the following holds.

Theorem. *If conditions a), b), c) are satisfied, the linear differential game (1) can be terminated on some set $A \subset R$ in a completely definite computable time $T(z_0)$.*

Proof is carried out by reduction to Theorem 1 of [1], which concerns general, and not only linear, differential games. The set A and the termination time of the game are determined according to the prescription indicated in [1]. Here we shall carry out the reduction itself, and also indicate the condition under which $A = R$.

The function $\omega(s)$ (see [1]) for the linear game (1) is written explicitly, namely

$$z = \omega(s) = \omega(\tau, \varphi, \xi) = e^{\tau C}\left[\xi + \int_0^\tau e^{-rC}\left(U(u(r, \varphi)) - V(v(r, \varphi))\right) dr\right], \tag{6}$$

where $s = (s^1, s^2, \dots, s^n)$; $\tau = s^1$; $\varphi = (s^2, \dots, s^\nu)$ is an arbitrary point of the sphere K; $\xi = (s^{\nu+1}, \dots, s^n)$ is an arbitrary point of the subspace M; $u(\tau, \varphi), v(\tau, \varphi)$ are the functions occurring in the mapping (5).

The linear independence of the vectors $\partial\omega/\partial s^2, \dots, \partial\omega/\partial s^n$ is equivalent to the local convexity of the surface Σ_τ, and thus condition 1 of [1] is satisfied in our case. Condition 2 for game (1) can easily be verified in the most general case. In particular, it follows from the stronger condition c). The remaining conditions 3, 4, 6, 8, 9 of [1] for the linear game follow almost automatically from conditions a), b).

We give here the proof that, under the assumptions of the theorem formulated above, the condition of [1] which is most difficult to verify is satisfied, namely condition 5: that for any point s belonging to the upper layer of the mapping $\omega(s)$, the function $H(s)$ is nonnegative. At the final stage of the proof we shall use the following auxiliary proposition, which is contained in § 7 of [1]:

Lemma. *Let the equation*

$$F(\tau, z) = 0, \qquad z \notin M, \tag{7}$$

be obtained as a result of eliminating the variables $s^2, \dots, s^n$ from equation (6) (this means that if a point z satisfies equation (6), then the negative number τ satisfies equation (7), and conversely, if the negative number τ satisfies equation (7), then there exists a point $s = (\tau, \varphi, \xi)$ satisfying equation (6)). Then the following two relations hold:

$$\partial F(\tau, \omega(s))/\partial z = -b(s)\varphi e^{-\tau C}; \qquad F'_\tau(\tau, \omega(s)) = b(s)H(s), \quad b(s) \neq 0. \tag{8}$$

We construct the function $F(\tau, z)$ for game (1). To this end consider the mapping

$$y = \int_0^\tau w(r, \varphi)\, dr, \tag{9}$$

obtained by integration with respect to τ of the mapping (5). The image of the sphere K under the mapping (9) is the locally convex surface Δ_τ, and since τ is negative, the vector φ is its inward normal at the corresponding point. It can be proved that Δ_τ is convex as a whole. For simplicity of construction, let us assume that Δ_τ contains the origin of coordinates in its interior. We now construct the auxiliary function $\lambda_\tau(y)$, defined for every point $y \in L$ as follows. Let the ray Oy intersect

surface Δ_τ at the point y_0. Put

$$\lambda_\tau(y) = |Oy|/|Oy_0|. \tag{10}$$

Then the equation of the surface Δ_τ is written, obviously, in the form $\lambda_\tau(y) - 1 = 0$, and the function $F(\tau, z)$ in the form

$$F(\tau, z) = \lambda_\tau(\pi e^{-\tau C} z) - 1. \tag{11}$$

Since the vector $\partial\lambda_\tau(y)/\partial y$ is the outer normal to the surface Δ_τ, while the vector φ is the inner normal, it follows that $\partial\lambda_\tau(y)/\partial y = -g(y)\varphi$, where $g(y) > 0$. Therefore

$$\frac{\partial F}{\partial z} = \frac{\partial \lambda_\tau(y)}{\partial y}\frac{dy}{dz} = -g(y)\varphi e^{-\tau C}. \tag{12}$$

Comparing this relation with the first of relations (8), we obtain $g(\pi e^{-\tau C}\omega(s)) = b(s)$. Consequently, by virtue of the second relation (8),

$$F'_\tau(\tau, \omega(s)) = g(s)H(s), \qquad g(s) > 0. \tag{13}$$

It follows from this that the function $H(s)$ has at each point s the same sign as $F'_\tau(\tau, \omega(s))$.

Let now the point s belong to the upper sheet of the mapping (6), and let $z = \omega(s)$ be its image. Then the number τ is the root of least absolute value of equation (7). On the other hand, from the definition of the function $\lambda_\tau(y)$ it follows that $F(0, z) = +\infty$. Therefore, when passing through the greatest negative root, $F(\tau, z)$ increases and, thus, $F'_\tau(\tau, z) \geq 0$, and hence also $H(s) \geq 0$, as was required to prove.

3. In order that the set A, on which game (1) can be ended, coincide with the whole space R, it is sufficient that, under the mapping ω, every point $z \in R$ have a preimage $s \in S$. By the lemma of the preceding paragraph, for this it is, in turn, sufficient that for any $z \in R$ equation (7) have at least one negative root.

It is not hard to see that this condition is fulfilled if, for each point $z \in R$, there exists at least one negative value τ such that the point $\pi e^{-\tau C} z$ lies inside the surface Δ_τ. Verification of this property can easily be carried out by letting τ tend to $-\infty$.

4. In proving the theorem of item 2, we in fact used the local convexity of the surface Δ_τ, which follows directly from condition b). However, even in the case when the surface Δ_τ is not locally convex, a considerable part of the construction can be preserved. Namely, if the surfaces

$$y = \pi \int_0^\tau e^{-rC} U(u(r, \varphi))\, dr, \qquad y = \pi \int_0^\tau e^{-rC} V(v(r, \varphi))\, dr$$

possess the property that the second of them can be carried by a translation into the interior of the first, then from the surface Δ_τ one can extract a convex, piecewise locally convex part Δ_τ^*, bounding a certain convex domain in L. The surface Δ_τ^* can then be taken as the basis for constructing the function $\lambda_\tau(y)$.

The extraction of the convex part Δ_τ^* of the surface Δ_τ means that, in considering game (1), we use not the whole manifold S, but only a certain part S^* of it, which is obtained if one takes those points (τ, φ, ξ) for which

$$\int_0^\tau w(r, \varphi)\, dr \in \Delta_\tau^*,$$

and then performs the gluing. However, the question of passage through the gluing surfaces requires special study.

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1. L. S. Pontryagin, UMN, 21, no. 4 (130), 219 (1966).

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