HOMOTOPIES OF COMPLEMENTS TO KNOTS

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HOMOTOPIES OF COMPLEMENTS TO KNOTS

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In this paper we study the homotopy properties of complements to finite and infinite connected sums of locally flat spheres of codimension 2 in Euclidean space R^n for $n \geq 5$. As applications, simple proofs and strengthenings are obtained for some results announced in note (^{1}).

1. Definitions and notation.

Locally flat embeddings of a sphere of codimension 2 will be written in the form $\sigma : S \subset R^n$ and called **knotted** if there is no homeomorphism of R^n onto itself carrying S to the standard sphere. If $\sigma_i : S_i \subset R^n$, $i = 1, 2$, are two such embeddings, and the spheres S_1 and S_2 lie on different sides of a hyperplane $R^{n-1} \subset R^n$, intersect in a rectilinear simplex

$$\Delta = S_1 \cap S_2 \subset R^{n-1}$$

and are such that

$$\overline{S_1 \cup S_2 \setminus \Delta}$$

is transversal to R^{n-1}, then the embedding

$$\overline{S_1 \cup S_2 \setminus \Delta} \subset R^n$$

is called the **connected sum** (or simply the sum) of the embeddings σ_1 and σ_2, and is denoted by

$$\sigma_1 \# \sigma_2 : \overline{S_1 \cup S_2 \setminus \Delta} \subset R^n.$$

Let

$$H_k\sigma = H_k(R^n \setminus S)$$

and

$$\pi_k\sigma = \pi_k(R^n \setminus S).$$

If $k \geq 2$, then on the group $\pi_k\sigma$ there acts (as a ring of operators) the group ring $Z[\pi_1\sigma]$ of the fundamental group; the module obtained as a result of this action will be denoted by $\tilde{\pi}_k\sigma$.

We shall call **meridians** those elements of the group $\pi_1\sigma$ which, under the homomorphism of commutation

$$\pi_1\sigma \to H_1\sigma \simeq Z,$$

are mapped to the element $+1$ of the additive group of integers Z. All meridians are mutually conjugate. Therefore, when choosing a minimal system of generators of the group $\pi_1\sigma$, one may take as the first generator some meridian μ, and choose the remaining generators $\{v_\alpha\}$ so that their linking coefficients with S are equal to zero. When summing two spheres, we shall agree to choose the meridians of the summands so that they are homotopic to one another in the complement to the summed sphere. Then, after the natural identifications, one may assume that the fundamental groups of the summands σ_1 and σ_2 have a common subgroup

$$M \simeq Z,$$

generated by the meridian μ. It is known that then the fundamental group of the sum is the free product of the fundamental groups of the summands with the identified (amalgamated) subgroup M; this is written in the form

$$\pi_1(\sigma_1 \# \sigma_2) \simeq \pi_1\sigma_1 *_M \pi_1\sigma_2.$$

2. Homotopy groups of locally flat spheres.

Theorem 1. *For any locally flat sphere $\sigma : S \subset R^n$, the fundamental group $\pi_1\sigma$ has a finite number of generators.*

The assertion analogous to Theorem 1 for the homotopy groups $\pi_k\sigma$ with $k \geq 2$ is false: the point is that $\pi_1\sigma$ acts on $\pi_k\sigma$, and each element of infinite order $v \in \pi_1\sigma$, together with nonzero elements $\theta \in \pi_k\sigma$, generates a countable set

$$A = \{v^i\theta\}$$

of elements of the group $\pi_k\sigma$, between which, generally speaking, there are no additive relations. But if one considers the set A as a subset of the module $\tilde{\pi}_k\sigma$, then A is generated by one

generator θ. The author does not know, however, whether the module $\tilde{\pi}_k\sigma$ will always be finitely generated; apparently this is false, but it is so in the case when $\pi_1\sigma \cong Z$. Indeed, the following holds.

Theorem 2. If a locally flat sphere $\sigma : S \subset R^n$ satisfies the relations $\pi_1\sigma \cong Z$, $\pi_2\sigma = \cdots = \pi_{k-1}\sigma = 0 \neq \pi_k\sigma$, $n \geq 5$, then the module $\pi_k\sigma$ has a finite number of (modular) generators*.

We briefly describe the proofs of Theorems 1 and 2. One may assume that $S \supset \Delta$, where Δ is an $(n-2)$-dimensional rectilinear disk. Let B be a geometric ball not intersecting $S \setminus \Delta$ but intersecting Δ, and let $D = S \setminus B$. Choose a neighborhood U of the interior of the disk D, homeomorphic to the direct product $R^{n-2} \times R^2$. Take a subdivision of the triangulation of R^n so fine that the closed star $\mathrm{St}(S \setminus \mathrm{Int}\,\Delta)$ (i.e. the set of all closed simplexes of R^n intersecting $S \setminus \mathrm{Int}\,\Delta$) is entirely contained in U. Let $P = \Delta \cup \mathrm{St}(S \setminus \mathrm{Int}\,\Delta)$ and $P' = R^n \setminus RP$, where RP denotes the regular neighborhood, in the sense of Whitehead (^{3}), of the polyhedron P. Consider the inclusions $R^n \setminus S \supset R^n \setminus P \supset R^n \setminus (S \cup U)$ and the diagram generated by them

$$\begin{tikzcd} \pi_1\sigma = \pi_1(R^n \setminus S) \arrow[dr] & & \pi_1(R^n \setminus S \cup U) \arrow[ll] \arrow[dl] \\ & \pi_1(R^n \setminus P) \cong \pi_1 P' & \end{tikzcd} \tag{*}$$

In this diagram the upper row is an isomorphism (since $R^n \setminus S$ contracts onto $R^n \setminus (S \cup U)$). The group $\pi_1(R^n \setminus P)$ is finite, since P' is a finite polyhedron. Hence, by the commutativity of the diagram, Theorem 1 follows.

To prove Theorem 2 it is enough to construct the same diagram for $\tilde{\pi}_k$ (instead of π_1). But commutativity for the modules $\tilde{\pi}_k$ does not follow at once, since the group $\pi_1(R^n \setminus P)$ need not be isomorphic to Z. However, without going beyond U, one can reconstruct P into a polyhedron $\tilde{P}$ so that $\pi_1(R^n \setminus \tilde{P}) \cong Z$ (see, for example, the analogous arguments in Browder' s paper (^{4}), Lemma 3.1). Then the triangular diagram for $\tilde{\pi}_k$, analogous to $(*)$, is commutative, the upper row is an isomorphism of modules, and Theorem 2 will be proved if we establish finite generation of the module $\tilde{\pi}_k(R^n \setminus \tilde{P})$. But this last fact follows from the following lemma.

Lemma 1. If Q is a finite polyhedron and $\pi_1 Q \cong Z$, $\pi_2 Q = \cdots = \pi_{k-1} Q = 0 \neq \pi_{kQ}$, then the module $\tilde{\pi}_{kQ}$ is finitely generated.

This lemma is proved by means of the Hurewicz theorem by passing to the universal covering polyhedron Q.

3. Homotopy groups of the sum of two spheres. It was noted above that $\pi_1(\sigma_1 \# \sigma_2) \cong \pi_1\sigma_1 *_M \pi_1\sigma_2$, where M is the subgroup generated by a meridian. For what follows it is essential to know how the generators of the groups $\pi_1\sigma_i$ behave under amalgamation. It turns out that if minimal systems of generators are chosen as indicated in §1, then a minimal system of generators of the group $\pi_1(\sigma_1 \# \sigma_2)$ can be obtained by adjoining the meridian and all non-meridional generators of both groups $\pi_1\sigma_1$ and $\pi_1\sigma_2$. This follows from the following purely algebraic lemma.

Lemma 2. Let G_1 and G_2 be groups with a finite number of generators, containing a free subgroup $M = G_1 \cap G_2$ with one generator μ. Let $\mu, \lambda_1, \dots, \lambda_{p_1-1}$ and $\mu, \nu_1, \dots, \nu_{p_2-1}$ be minimal systems of generators for G_1 and G_2. Let $\mu, \beta_1, \dots, \beta_p$ be a minimal system of generators of the group $G_1 *_M G_2$ among all systems of generators of this group of the form

$$\mu, \tau_1, \dots, \tau_m. \tag{**}$$

Then $p = p_1 + p_2 - 1$, and as such a minimal system one may take $\mu, \lambda_1, \dots, \lambda_{p_1-1}, \nu_1, \dots, \nu_{p_2-1}$.**

* Examples of locally flat (even smooth) spheres with the indicated properties can be found in Kervaire' s paper (^{2}).

** The author proposes to give a generalization of this lemma in another paper.

The proof of this lemma is similar to the proof of Grushko' s theorem due to Stallings (^{5}). The groups G_1 and G_2 are realized as fundamental groups of two-dimensional complexes A_1 and A_2, spanned by bouquets of circles $m, l_1, \dots, l_{p_1-1}$ and $m, n_1, \dots, n_{p_2-1}$, where the common circle $m = A_1 \cap A_2$ realizes the generator μ. The choice of a minimal system of generators of the form $(**)$ is equivalent to the construction of a continuous mapping
$f : B \to A = A_1 \cup A_2$ with the following properties: B is a bouquet of p circles $\bar{m}, \bar{r}_1, \dots, \bar{r}_{p-1}$; $f : \bar{m} \to m$ has degree 1; $f_* : \pi_1 B \to \pi_1 A$ is an epimorphism.

For the proof of Lemma 2 we must reconstruct the mapping f into a mapping $f' : B \to A$ in such a way that f' sends each circle of the bouquet B either entirely into A_1 or entirely into A_2, while preserving the epimorphicity of the induced mapping of fundamental groups. For this, take coverings $\varphi : \widetilde{A} \to A$ and $\varphi : \widetilde{B} \to B$ of the complexes A and B, corresponding to the subgroups of the groups $\pi_1 A$ and $\pi_1 B$ generated by nonmeridional generators. Then, if $\widetilde{A}_i = \varphi^{-1} A_i$, $i = 1, 2$, the set $\widetilde{A}_1 \cap \widetilde{A}_2 = \varphi^{-1} m$ is homeomorphic to a line and therefore is homotopically trivial (unlike the circle $m = A_1 \cap A_2$!). In the complexes $\widetilde{A}$ and $\widetilde{B}$ there act, respectively, the groups M and $\overline{M}$ (generated by the elements $\mu \in \pi_1 A$ and $f_*^{-1} \mu \in \pi_1 B$) as groups of translations along the lines $\varphi^{-1} m$ and $\varphi^{-1} \bar{m}$. Consequently, the groups M and $\overline{M}$ act respectively in the fundamental groups $\pi_1 \widetilde{A}$ and $\pi_1 \widetilde{B}$, and the factorization by the action of M and $\overline{M}$ is equivalent to the mappings φ and φ.

One can cover the mapping $f : B \to A$ by a mapping $\tilde{f} : \widetilde{B} \to \widetilde{A}$ so that $\tilde{f}_* : \pi_1 \widetilde{B} \to \pi_1 \widetilde{A}$ is an epimorphism and the resulting quadrilateral diagram $(***)$ is commutative (the action of the groups M and $\overline{M}$ in $\pi_1 \widetilde{B}$ and $\pi_1 \widetilde{A}$ is compatible with the mapping $\tilde{f}_*$).

Now, using the technique of Stallings (^{5}), we reconstruct the mapping $\tilde{f}$ into a mapping $\tilde{f}' : \widetilde{B}' \to \widetilde{A}$ with the following properties: 1) the complex $\widetilde{B}'$ is homotopy equivalent to $\widetilde{B}$; 2) the complex $\widetilde{B}'$ decomposes into a sum $\widetilde{B}' = \widetilde{B}'_1 \cup \widetilde{B}'_2$, where $\tilde{f}' \widetilde{B}'_i = \widetilde{A}_i$. (Here it is essential that $\widetilde{A}_1 \cap \widetilde{A}_2$ is homotopically trivial: Stallings' reconstructions are not applicable to $f : B \to A$, since $A_1 \cap A_2 = m$ is a circle.)

Finally, using the diagram $(***)$ and factoring the mapping $\tilde{f}' : \widetilde{B}' \to \widetilde{A}$ by the action of M and $\overline{M}$, we obtain the desired mapping $f' : B \to A$, and therefore Lemma 2.

It follows from this lemma that, when a nontrivial locally flat sphere is summed with a given locally flat sphere, the minimum number of generators of the fundamental group increases by at least one.

Lemma 3. *If two locally flat spheres $\sigma_i : S_i \subset R^n$, $i = 1, 2$, are given such that $\pi_1\sigma_i \cong Z$, $\pi_2\sigma_i = \dots = \pi_{k-1}\sigma_i = 0 \neq \pi_k\sigma_i$, $i = 1, 2$, then the direct-sum formula holds*

$$\pi_k(\sigma_1 \# \sigma_2) = \pi_k\sigma_1 \oplus \pi_k\sigma_2.$$

This lemma is also proved by passing to the universal covering. In addition to the Hurewicz theorem, one also applies here the Mayer–Vietoris additive exact sequence.

4. Simple decompositions of locally flat spheres.

Let a locally flat sphere $\sigma : S \subset R^n$ be given. We shall say that this sphere is represented as a sum $\sigma_1 \cong \sigma_2$ if the embedding $\sigma_1 \# \sigma_2 : S' \subset R^n$ is equivalent to the embedding $S \subset R^n$ (i.e., there exists a homeomorphism of R^n onto itself taking S to S'). A locally flat sphere is called **simple** if it cannot be represented as a sum of knotted spheres.

Theorem 3. *Every locally flat sphere $\sigma : S \subset R^n$, for $n \geq 5$, is representable as a finite sum of simple spheres.*

If the theorem were false, then we would have an infinite sequence of equalities:

$$\sigma = \sigma_1 \# \sigma_2', \quad \sigma = \sigma_1 \# \sigma_2 \# \sigma_3', \dots, \quad \sigma = \sigma_1 \# \dots \# \sigma_k \# \sigma_{k+1}', \dots \qquad (\#)$$

where all σ_i are knotted spheres.

Let q be the least number such that among the groups $\pi_q\sigma_i$ there are infinitely many distinct from $\pi_q S^1$. That such a number exists (it is less than $n/2$) can be proved by using Stallings' unknotting criterion.

In the case $q = 1$, a contradiction with Theorem 1 is obtained by counting the number of generators of the group $\pi_1\sigma$ (using (#) and Lemma 2).

The case $q > 1$ is somewhat more complicated. Without loss of generality, one may assume that for all i, $\pi_l\sigma_i \cong \pi_l S^1$ $(l > q)$, and $\pi_q\sigma_i \neq 0$. Let $\sigma : S \subset R^n$, $\sigma_i : S_i \subset R^n$, $B = R^n \setminus S$, $B_i = R^n \setminus S_i$, and let $\tilde{B}, \tilde{B}_i$ be the universal covering spaces of B, B_i. By assumption, the fundamental groups of the knots σ, σ_i are isomorphic to the group of integers; we shall write this group in the form $J = \{t^m\}$ and also interpret it as the monodromy group in the covering spaces. From the Hurewicz theorem one obtains isomorphisms of $Z[J]$-modules $\pi_q\sigma \cong H_q\tilde{B}$, $\pi_q\sigma_i \cong H_q\tilde{B}_i$. Denote $A = H_q\tilde{B}$, $A_i = H_q\tilde{B}_i$, ${}_0A = H_q(\tilde{B}; Q)$, ${}_0A_i = H_q(\tilde{B}_i; Q)$, where Q is the field of rational numbers. The groups ${}_0A, {}_0A_i$

are naturally endowed with the structure of $Q[J]$-modules, and the rational group ring $Q[J]$ of the group J is a principal ideal ring.

This last fact, together with the theorem on the canonical decomposition of modules over a principal ideal ring, makes it possible to obtain a contradiction in the case when ${}_0A_i \neq 0$ for an infinite set of indices i. Indeed, for ${}_0A_i$ there is a sum formula analogous to Lemma 3, and counting the generators of the module ${}_0A$ from the relations (#) leads to a contradiction with Theorem 2.

Finally, in the case when ${}_0A_i = 0$ only for a finite number of indices i, one may, without loss of generality, assume that ${}_0A_i = 0$ for all i. It follows that all elements of ${}_0A$ and ${}_0A_i$ are of finite order. Next, a computation of the Cartan spectral sequence shows that the factor module A_J under the action of J is trivial. Hence, from Kervaire' s lemma ((7), item 8) we conclude that the number of elements of finite order in A and A_i is finite. A contradiction with Theorem 2 is then obtained immediately from Lemma 3 and the relations (#).

5. Infinite sums. If an infinite collection of knots is given, then one can naturally define the infinite sum of these knots by summing successively and shrinking to a single limit point (see (1)).

Theorem 4. *The infinite sum of spheres $\sigma_i : S_i \subset R^n$ is not locally flat at the limit point if among the summands there are infinitely many knotted ones and $n \geq 5$.*

The theorem is proved in the same way as Theorem 3.

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[*] In [1] there is a hint at a proof of this theorem, reducing to the fact that the theorem follows from counting the fundamental group of the sum sphere. This assertion is erroneous, since one can give examples of non-locally flat spheres with fundamental group isomorphic to Z.

Note: Figure translations are in progress. See original paper for figures.

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