ON A SYSTEM OF BOUNDARY-LAYER EQUATIONS FOR AXISYMMETRIC FLOWS

HYDROMECHANICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.945.7 : 532.526.2

HYDROMECHANICS

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ON A SYSTEM OF BOUNDARY-LAYER EQUATIONS FOR AXISYMMETRIC FLOWS

(Presented by Academician I. G. Petrovskii on May 7, 1967)

We shall consider the system of boundary-layer equations for axial stationary and nonstationary flow past bodies of revolution. For these systems, solutions will be constructed below by means of the method of lines; theorems on uniqueness and stability of solutions will be obtained; and the question of the behavior of solutions as time increases without bound will be investigated. In paper (^{1}), by the method of lines, solutions were constructed for the system of boundary-layer equations for nonstationary axisymmetric and plane-parallel symmetric flows over some finite time interval. Problems concerning continuation of the boundary layer for stationary and nonstationary flows were studied in papers ($^{1-3}$). We note that the proof of the existence of a solution of the Prandtl system that will be indicated below contains an approximate method for constructing the solution and a proof of its convergence. The method of A. A. Dorodnitsyn for the approximate solution of the system of boundary-layer equations is set forth in (^{4},^{5}).

1. Stationary axisymmetric boundary layer

The system of boundary-layer equations for an axisymmetric three-dimensional stationary flow of an incompressible fluid in the corresponding coordinates has the form

$$uu_x + vu_y = UU_x + \nu u_{yy}, \qquad (ru)_x + (rv)_y = 0 \tag{1}$$

in the domain $D_\theta\{0 \le x \le \theta,\ 0 \le y < \infty\}$, under the conditions

$$u|_{x=0} = 0, \qquad u|_{y=0} = 0, \qquad v|_{y=0} = v_0(x), \qquad u \to U(x) \quad \text{as } y \to \infty. \tag{2}$$

The function $r(x)$ determines the surface of the body being flowed around; $r(0) = 0$; $r_x(0) \neq 0$; U is the velocity of the external flow; ν is the coefficient of viscosity (see (^{6}), p. 174).

If one introduces new independent variables

$$\xi = x, \qquad \eta = u(x,y)/U(x), \tag{3}$$

then for the function $w = u_y/U$ we obtain the equation

$$\nu w^2 w_{\eta\eta} - \eta U w_\xi + A w_\eta + B w = 0 \tag{4}$$

in the domain $G_\theta\{0 \le \xi \le \theta,\ 0 \le \eta \le 1\}$, with conditions

$$w|_{\eta=1} = 0, \qquad (\nu w w_\eta - v_0 w + C)|_{\eta=0} = 0, \tag{5}$$

where $A = (\eta^2 - 1)U_x,\ B = -\eta U_x + \eta r_x U/r,\ C = U_x$.

Using the method of lines, we shall prove, under certain natural assumptions, the existence and uniqueness of a solution of problem (4), (5), which, through the transformation (3), leads to a solution of problem (1), (2). Let $f^k(\eta)$ denote $f(kh,\eta)$; $h = \text{const} > 0$; $k = 0, 1, 2, \ldots$. We replace equation (4) with conditions (5) by the system of ordinary differential equations

$$\nu(w^k)^2 w^k_{\eta\eta} - \eta U^k(w^k - w^{k-1})/h + A^k w^k_\eta + B^k w^k = 0, \tag{6}$$

$0 \leqslant \eta \leqslant 1;\ k = 0, 1, \ldots, [\theta/h]$, with the conditions

$$w^k(1) = 0, \qquad \left(\nu w^k w^k_\eta - v^k_0 w^k + C^k\right)\big|_{\eta=0} = 0. \tag{7}$$

We shall assume that U and r are twice continuously differentiable functions; $U(x) > 0$, $r(x) > 0$ for $x > 0$; $U = ax + b(x)$, where $a = \text{const} > 0$; $|b| \leqslant K_1 x^2$; v_0, v_{0x}, B_ξ are bounded, $U_x > 0$. It is obvious that $|B| \leqslant K_2\xi$; $K_1, K_2 = \text{const}$.

Lemma 1. The system (6), (7) has a solution $w^k(\eta)$, continuous for $0 \leqslant \eta \leqslant 1$ and possessing all derivatives for $\eta < 1$.

Lemma 2. For the solution of the problem (6), (7) the estimate

$$M(1-\eta)[|\ln(1-\eta)\beta_1|^{1/2} - M_0]e^{-\gamma_1 kh} \leqslant w^k(\eta) \leqslant M(1-\eta)|\ln(1-\eta)\beta_2|^{1/2} e^{\gamma_2 kh}, \tag{8}$$

holds, where $M^2 = 4a$; $\gamma_1, \gamma_2, M_0, \beta_1, \beta_2$ are certain positive constants independent of h, and $|\ln(1-\eta)\beta_1|^{1/2} - M_0 > 0$.

Lemma 3. For $kh \leqslant x_0$, where $x_0 > 0$ depends on U, r, v_0, the following estimates are valid for the solution $w^k(\eta)$ of the system (6), (7):

$$-M_1|\ln(1-\eta)\beta_2|^{1/2} \leqslant w_\eta^k(\eta) \leqslant -M_2|\ln(1-\eta)\beta_1|^{1/2}, \tag{9}$$

$$\left|(w^k - w^{k-1})/h\right| \leqslant M_3(1-\eta)|\ln(1-\eta)\beta_2|^{1/2}, \qquad |w^k w_{\eta\eta}^k| \leqslant M_4, \tag{10}$$

where the constants M_i are positive and do not depend on h.

Estimates for $w^k = z^k$ and $(w^k - w^{k-1})/h = r^k$ can be obtained by considering successively the equations

$$\nu(w^k)^2 z_\eta^k + A^k z^k = \eta U^k r^k - B^k w^k, \tag{11}$$

$$\nu(w^{k+1})^2 r_{\eta\eta}^{k+1} - \eta U^k(r^{k+1} - r^k)/h + A^{k+1} r_\eta^{k+1} + r^{k+1}\left[B^{k+1} - \eta(U^{k+1} - U^k)/h + \right.$$

$$\left. + (w^{k+1} + w^k)(\eta U^k r^k - A^k z^k - B^k w^k)/(w^k)^2\right] + w_\eta^k(A^{k+1} - A^k)/h + w^k(B^{k+1} - B^k)/h = 0 \tag{12}$$

for $k = 0$, then for $k = 1, 2, \dots, [\theta/h]$, and taking into account the boundary conditions: $r^{k+1}(1) = 0$; $\nu r_\eta^{k+1} - C^{k+1} r^{k+1}/w^k w^{k+1} + (C^{k+1} - C^k)/w^k h + (v_0^{k+1} - v_0^k)/h = 0$ at $\eta = 0$; $\nu z^k - v_0 + C^k/w^k = 0$ at $\eta = 0$, as well as the inequalities (8), from which estimates of the form (9) follow for z^k for some sequence of points η_n tending to $\eta = 1$ as $n \to \infty$. Estimate (10) for $w^k w_{\eta\eta}^k$ follows directly from equations (6) and the preceding inequalities. By Lemmas 1 and 2, the functions w^k form a compact family in the sense of uniform convergence. The limiting function w as $h \to 0$ is continuous in G_{x_0} and has a bounded derivative w_ξ; the derivatives w_η, $w_{\eta\eta}$ are such that ww_η, $ww_{\eta\eta}$ are bounded. Since at the interior points of G_{x_0} equation (4) is parabolic, the derivatives w_ξ, w_η, $w_{\eta\eta}$ satisfy the Hölder condition in any interior subdomain G_{x_0}, and within G_{x_0} equation (4) is valid for w. By virtue of the boundedness of $w_{\eta\eta}^k$ for $\eta < 1 - \delta$ and any $\delta > 0$, w_η satisfies a Lipschitz condition in η for $\eta < 1 - \delta$, and condition (5) is fulfilled for $\eta = 0$. Hence we obtain the following assertion.

Theorem 1. If $\theta \leqslant x_0$, then in D_θ there exists a solution of the problem (1), (2) possessing the following properties: u, u_y are continuous and bounded in D_θ; u_{yy}, u_x, v_y are bounded and continuous in y in D_θ and continuous in x at the interior points of D_θ; v is continuous in y in D_θ, in x at the interior points of D_θ, and bounded for bounded y; $u_y \to 0$ as $y \to \infty$, $u_y > 0$ for $0 \leqslant y < \infty$; u_{yyy} is bounded in D_θ; u_{yx} is bounded in D_θ for bounded y; u_{yx}, u_{yyy} are continuous at the interior points of D_θ. The solution of the problem (1), (2) with the indicated properties is unique.

2. Nonstationary axisymmetric boundary layer. The Prandtl system for an axisymmetric three-dimensional unof the steady flow (see (6))

$$u_t + uu_x + vu_y = UU_x + U_t + \nu u_{yy}, \qquad (ru)_x + (rv)_y = 0 \tag{13}$$

in the domain $Q_\theta\{0 \le t < \infty,\ 0 \le x \le \theta,\ 0 \le y < \infty\}$, with the conditions

$$u\big|_{t=0} = u_0(x,y), \qquad u\big|_{x=0} = 0, \qquad u\big|_{y=0} = 0, \qquad v\big|_{y=0} = v_0(t,x),$$

$$u \to U(t,x) \quad \text{as } y \to \infty, \tag{14}$$

like problem (1), (2), by means of the transformation of the independent variables

$$\tau = t, \qquad \xi = x, \qquad \eta = u(t,x,y)/U(t,x) \tag{15}$$

and the introduction of the new function $w = u_y/U$, is reduced to the equation

$$\nu w^2 w_{\eta\eta} - w_\tau - \eta U w_\xi + \widetilde{A} w_\eta + \widetilde{B} w = 0 \tag{16}$$

in the domain $\Omega_\theta\{0 \le \tau < \infty,\ 0 \le \xi \le \theta,\ 0 \le \eta \le 1\}$, with the conditions

$$w\big|_{\tau=0} = u_{0y}/U(0,x) \equiv w_0, \quad w\big|_{\eta=1} = 0, \quad (\nu w w_\eta - v_0 w + \widetilde{C})\big|_{\eta=1} = 0, \tag{17}$$

where

$$\widetilde{A} = (\eta^2 - 1)U_x + (\eta - 1)U_t/U, \qquad \widetilde{B} = \eta r_{xU}/r - U_t/U - \eta U_x,$$

$$\widetilde{C} = U_x + U_t/U.$$

We shall assume that $r(t,x)$ and $U(t,x)$ are functions twice continuously differentiable in Q_θ; $U > 0$ and $r > 0$ for $x > 0$; $U = ax + b(t,x)$, where $a = \text{const} > 0$, $|b| \le K_3 x^2$, $K_3 = \text{const}$; $r(t,0) = 0$, $r_x(t,0) \ne 0$; $v_0(t,0) = \text{const}$; the functions U_x, U_t/U, r_{xU}/r, v_0 and their derivatives with respect to t and x are bounded in Q_θ, and the derivatives of these functions with respect to t, as well as U_t/U, do not exceed in absolute value $K_4 x$, where $K_4 = \text{const}$.

Let $f^{mk}(\eta) \equiv f(mh, kh, \eta)$. We replace equation (16) in the domain Ω_θ, with conditions (17), by a system of ordinary differential equations

$$\nu(w^{mk})^2 w^{mk}_{\eta\eta} - \frac{w^{mk}-w^{m-1\,k}}{h} - \eta U^{mk}\frac{w^{mk}-w^{mk-1}}{h} + \widetilde{A}^{mk} w^{mk}_{\eta} + \widetilde{B}^{mk} w^{mk} = 0, \tag{18}$$

$$0 \le \eta \le 1; \qquad m = 1, 2, \dots; \qquad k = 0, 1, \dots, [\theta/h]; \qquad h = \mathrm{const} > 0,$$

with the conditions

$$w^{mk}(1) = 0, \qquad \left(\nu w^{mk} w^{mk}_{\eta} - v_0^{mk} w^{mk} + \widetilde{C}^{mk}\right)\Big|_{\eta=0} = 0, \qquad w^{0k} = w_0(kh, \eta). \tag{19}$$

With respect to the function $w_0(\xi,\eta)$ we shall assume the following: $w_0 \le M(1-\eta)|\ln(1-\eta)\alpha_1|^{1/2}e^{\gamma\xi}$, where $M^2 = 4a$; $w_0 \ge K_4(1-\eta)|\ln(1-\eta)\alpha_2|^{1/2}e^{-\gamma\xi}$, moreover $K_4^2 \ge 2a$, $|w_{0\xi}| \le K_5(1-\eta)|\ln(1-\eta)\alpha_2|^{1/2}$, if $|\widetilde{B}_{\xi}| \le K_6$, and K_4 is an arbitrary sufficiently small constant, while $|w_{0\xi}| \le K_7(1-\eta)\xi|\ln(1-\eta)\alpha_2|^{1/2}$, if $|\widetilde{A}_{\xi}| + |\widetilde{B}_{\xi}| + |\widetilde{C}_{\xi}| \le K_8\xi$; let, in addition, $w_{0\eta} \le 0$, $w_{0\eta} \ge -K_9|\ln(1-\eta)\alpha_1|^{1/2}$,

$$\left|\nu w_0^2 w_{0\eta\eta} - \eta U(0,\xi) w_{0\xi} + \widetilde{A}(0,\xi,\eta) w_{0\eta} + \widetilde{B}(0,\xi,\eta) w_0\right| \le K_{10}\xi(1-\eta)|\ln(1-\eta)\alpha_2|^{1/2}$$

and the compatibility condition is fulfilled

$$\nu w_0(\xi,0) w_{0\eta}(\xi,0) - v_0(0,\xi) w_0(\xi,0) + \widetilde{C}(0,\xi) \equiv \nu u_{0yy}(x,0) + (UU_x + U_t)\big|_{t=0} - v_0(0,x) u_{0y}(x,0) = 0.$$

Here K_i, α_i, γ are certain positive constants. Obviously, all these conditions on w_0 will be satisfied if, for example, $u_0(x,y)$ is a solution of problem (1), (2), or coincides with it in a neighborhood of the straight line $x = 0$, and also for large y, with U, r, v_0 taken at $t = 0$. This means that w_0 is a certain perturbation of the velocity u in the steady boundary layer.

Lemma 4. *The system (18) with conditions (19) has a solution w^{mk}, continuous for $0 \le \eta \le 1$ and possessing all continuous derivatives for $\eta < 1$. For $kh \le \tilde{x}_0$ and $0 \le mk < \infty$, the following estimates hold for w^{mk}:*

$$M_5(1-\eta)|\ln(1-\eta)\alpha_4|^{1/2} \le w^{mk} \le M(1-\eta)|\ln(1-\eta)\alpha_3|^{1/2}e^{\gamma kh},$$

$$0 \ge w^{mk}_{\eta} \ge -M_6|\ln(1-\eta)\alpha_3|^{1/2}, \qquad \left|\frac{w^{mk}-w^{m-1\,k}}{h}\right| \le$$

$$\leq M_7 kh(1-\eta)|\ln(1-\eta)\alpha_4|^{1/2}, \tag{20}$$

$$\left|\frac{w^{mk}-w^{mk-1}}{h}\right| \leq M_8(1-\eta)|\ln(1-\eta)\alpha_4|^{1/2}.$$

Here M_i and α_i are certain positive constants independent of h.

Theorem 2. Let U, r, v_0, u_0 satisfy the conditions indicated above in Sec. 2. Then in the domain Q_θ, for $\theta \leqslant \tilde{x}_0$, there exists a solution of problem (13), (14) having the following properties: u, u_y are continuous and bounded in Q_θ; u_{yy}, u_x, u_t, v_y are bounded and continuous with respect to y in Q_θ; v is continuous with respect to y and bounded for bounded y; $u_y \to 0$ as $y \to \infty$, $u_y > 0$ for $y \geqslant 0$; u_{yyy} is bounded in Q_θ, u_{yx}, u_{yt} are bounded in Q_θ for bounded y. The solution of problem (13), (14) possessing these properties is unique.

3. Stability. Denote by $\tilde{u}(t,x,y)$ the solution of problem (13), (14) for

$$U = \tilde{U}(t,x), \quad r = \tilde{r}(t,x), \quad v_0 = \tilde{v}_0(t,x)$$

and $u_0(x,y)$, and by $u(x,y)$ the solution of problem (1), (2) corresponding to the functions $U(x), r(x), v_0(x)$. Suppose that these functions satisfy the conditions of Theorems 2 and 1, respectively.

Theorem 3. Let the functions

$$\tilde{U}-U, \quad \tilde{U}_x - U, \quad \tilde{U}_t/\tilde{U}, \quad \tilde{v}_0 - v_0, \quad \tilde{r}_x\tilde{U}/\tilde{r} - r_{xU}/r \tag{21}$$

tend to zero as $t \to \infty$ uniformly in D_θ. Then for any $\delta > 0$ there exists a constant $C_1(\delta)$ such that in D_θ

$$\left|\tilde{u}(t,x,y)/\tilde{U}(t,x) - u(x,y)/U(x)\right| \leqslant \delta + C_1(\delta)e^{-\sigma t},$$

where $\sigma = \mathrm{const} > 0$ depends on the data of problem (1), (2), $\theta = \min(x_0, \tilde{x}_0)$.

Theorem 4. If the functions (21) are equal to zero for $t \geqslant t_0 > 0$, where $t_0 = \mathrm{const} < \infty$, then in D_θ, for $y \leqslant y_0$,

$$|\tilde{u}(t,x,y) - u(x,y)| \leqslant C_2 U(x) e^{-\sigma t},$$

where the constant C_2 depends on y_0, $\theta = \min(x_0, \tilde{x}_0)$.

Theorem 5. Let the functions (21) not exceed $\varepsilon > 0$ in absolute value, and let

$$|\tilde{w}(0,\xi,\eta) - w(\xi,\eta)| \leqslant \varepsilon,$$

where w is the solution of problem (4), (5), and $\tilde{w}$ is the solution of problem (16), (17), corresponding to the functions u and $\tilde{u}$. Then

$$\left|\tilde{u}(t,x,y)/\tilde{U}(t,x) - u(x,y)/U(x)\right| \leqslant C_3\varepsilon$$

for all t in D_θ, where the constant C_3 depends on the data of problem (1), (2) and on the constants entering into (20).

Let us note that, by an analogous method, one can investigate the boundary-layer equations for plane-parallel symmetric flows of an incompressible fluid. The question of the stability of solutions of the problem of the continuation of the boundary layer was considered in work (7).

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Received
6 V 1967

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Note: Figure translations are in progress. See original paper for figures.

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