ON THE EXISTENCE OF A BEST APPROXIMATION OF FUNCTIONS BY SUMS OF A FINITE NUMBER OF PLANE WAVES OF GIVEN DIRECTIONS IN THE METRIC \(L_p\)

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Abstract

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ON THE EXISTENCE OF A BEST APPROXIMATION OF FUNCTIONS BY SUMS OF A FINITE NUMBER OF PLANE WAVES OF GIVEN DIRECTIONS IN THE METRIC L_p

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Let O be an open set in the space $R_n = \{\mathbf{x} = (x_1, \dots, x_n)\}$; let $\varphi(\mathbf{x})$ be a finite basic function in O; let K_O be the space of all $\varphi(\mathbf{x})$; and let $\{f = (f, \varphi(\mathbf{x}))\}$ be the space of generalized functions on K_O.

We denote by $L_p(D)$, $D \subset R_n$ a bounded domain, the space of functions $f(\mathbf{x})$ for which $\|f(\mathbf{x})\|_p < \infty$, where

$$\|f(\mathbf{x})\|_p = \left(\int_D |f(\mathbf{x})|^p \, dx \right)^{1/p},$$

$$1 < p < \infty, \qquad |f(\mathbf{x})|_\infty = \inf_{M \subset D,\, \mathrm{mes}\, M = 0} \left\{ \sup_{\mathbf{x} \in D - M} |f(\mathbf{x})| \right\}.$$

Taking as given k distinct fixed points $\mathbf{a}_i$, $i = 1, \dots, k$, $\mathbf{a}_i \in \Pi_{n-1}$, $|\mathbf{a}_i| = 1$, where $\Pi_{n-1} = \{\mathbf{a} = (a_1, \dots, a_n)\}$ is the real projective space whose points are determined by homogeneous coordinates $a_1, \dots, a_n$, consider on each interval $\Lambda_i = (\alpha_i, \beta_i) = \{z_i : \ z_i = \mathbf{a}_i \mathbf{x} = a_{i1} x_1 + \dots + a_{in} x_n, \ \mathbf{x} \in D\}$ the class H_i of all functions $h_i(z_i)$ such that $h_i(\mathbf{a}_i \mathbf{x}) \in L_p^{\mathrm{loc}}(D)$. The symbol $L_p^{\mathrm{loc}}(D)$ denotes the space of all functions each of which belongs to $L_p(D')$ on any set D' compact in D.

Set

$$\rho_k = \inf \left\| f(\mathbf{x}) - \sum_{i=1}^{k} h_i(\mathbf{a}_i \mathbf{x}) \right\|_p, \tag{1}$$

where the infimum is taken over all $h_i \in H_i$, $\sum_{i=1}^{k} h_i(\mathbf{a}_i \mathbf{x}) \in L_p(D)$.

In this paper sufficient conditions are established for the existence of a system of functions $h_i^0(z_i)$, $h_i^0(\mathbf{a}_i\mathbf{x}) \in L_p^{\mathrm{loc}}(D)$, $i = 1, \dots, k$, for which the infimum (1) is attained; conditions are given under which $h_i^0(\mathbf{a}_i\mathbf{x}) \in L_p(D)$. At the same time it is shown that the quantity ρ_k is equal to the supremum of the moduli of the values of the generalized function

$$f = \int_D f(\mathbf{x})\varphi(\mathbf{x})\,dx$$

on a certain subset K_D.

Let $\mathbf{a} \in \Pi_{n-1}$, $|\mathbf{a}| = 1$, $\Delta = \{\mathbf{x} : \alpha < \mathbf{ax} < \beta\}$, where α and β $(-\infty \leq \alpha < \beta \leq +\infty)$ are fixed constants. A generalized function $(f, \varphi(\mathbf{x}))$, $\varphi(\mathbf{x}) \in K_\Delta$, will be called a **generalized plane wave of direction a** if, for every vector $\mathbf{b}$, $\mathbf{ab} = 0$, and for every function $\varphi(\mathbf{x}) \in K_\Delta$, the equality

$$(f, \varphi(\mathbf{x} + \mathbf{b})) = (f, \varphi(\mathbf{x}))$$

holds.

Suppose further that $h = (h, \psi(z))$ is a generalized function defined on the space $K_{(\alpha,\beta)}$ of basic functions of the variable z.

We shall call the superposition of h and the form $\mathbf{ax}$ a plane wave of direction $\mathbf{a}$, defined by the equality

$$f = \langle f, \varphi(\mathbf{x})\rangle = \left\langle h, \psi(z) = \int_{\mathbf{ax}=z} \varphi(\mathbf{x})\,ds \right\rangle,$$

where $\varphi(\mathbf{x}) \in K_\Delta$, and ds is the area element of the hyperplane $\mathbf{ax} = z$.

Lemma 1. *Every plane wave f of direction* $\mathbf{a}$ *is the superposition of some generalized function $h = \langle h, \psi(z)\rangle$, $\psi(z) \in K_{(\alpha,\beta)}$, and the form* $\mathbf{ax}$.

In proving this lemma one uses

Lemma 2. *Let $\mathbf{a}, \mathbf{b}_1, \dots, \mathbf{b}_{n-1}$, where $\mathbf{a}, \mathbf{b}_i \in R_n$, be an orthonormal frame. If, for $\varphi(\mathbf{x}) \in K_\Delta$,*

$$\int_{\mathbf{ax}=z} \varphi(\mathbf{x})\,ds = 0$$

for every z, then

$$\varphi(\mathbf{x}) = \partial F_1(\mathbf{x})/\partial b_1 + \dots + \partial F_{n-1}(\mathbf{x})/\partial b_{n-1}, \qquad F_i(\mathbf{x}) \in K_\Delta.$$

By Lemma 1, every generalized plane wave f of direction $\mathbf{a}$, by analogy with ordinary functions, can be written in the form $f = h(\mathbf{ax})$, and

$$\partial f/\partial x_i = \partial h(\mathbf{ax})/\partial x_i = h'_{\mathbf{ax}}(\mathbf{ax})a_i, \qquad i = 1, \dots, n.$$

The proof of the existence of an extremal system of functions $h_i^0(\mathbf{a}_i\mathbf{x})$, $i = 1, \dots, k$, is based on one lemma of functional analysis.

Let $E = \{e\}$ be a linear space of Banach type; E^* the conjugate space of all linear functionals $l(e)$ on E; $\Phi \subset E$ an arbitrary linear manifold; $U \subset E^*$ the set of linear functionals $U(e)$ that vanish on Φ.

Lemma 3 (S. Ya. Khavinson (^{1})). *For an arbitrary linear functional* $l(e)$*,*

$$\|l(e)\|_\Phi = \inf_{u \in U} \|l - u\|_{E^*}, \tag{2}$$

and there exists an element $u^0 \in U$ *for which the lower bound in (2) is attained. Here* $\|l\|_\Phi$ *is the norm of the functional* l *on the manifold* Φ*.*

The subsequent arguments rest on Theorem 1, which follows directly from Lemma 3.

Given a system of equations

$$P_i u = 0, \qquad i = 1, \dots, r, \tag{3}$$

where P_i is a homogeneous polynomial in $\partial/\partial x_1, \dots, \partial/\partial x_n$ with real constant coefficients, and u is an unknown generalized function on K_D.

Theorem 1. *Let* $f(\mathbf{x}) \in L_p(D)$*. Denote by* $U = \{u\}$ *the set of all functions* $u(\mathbf{x}) \in L_p(D)$*, each of which generates a regular generalized solution of system (3) on* K_D*. Then*

$$\inf_{u \in U} \|f(\mathbf{x}) - u(\mathbf{x})\|_p = \sup_{\|\widetilde{\varphi}\|_q = 1} \left| \int_D f(\mathbf{x}) \widetilde{\varphi}(\mathbf{x})\, d\mathbf{x} \right|, \tag{4}$$

where $\widetilde{\varphi}(\mathbf{x})$ *is an arbitrary function representable in the form*

$$\widetilde{\varphi} = P_1\varphi_1(\mathbf{x}) + \cdots + P_r\varphi_r(\mathbf{x}), \qquad \varphi_i(\mathbf{x}) \in K_D, \qquad 1/p + 1/q = 1.$$

Moreover, there exists $u^0(\mathbf{x}) \in U$ *for which the lower bound in (4) is attained.*

For the proof it suffices to set $E = L_q(D)$, $1/p + 1/q = 1$, $\Phi = \{\widetilde{\varphi}(\mathbf{x})\}$. In this case $E^* = L_p(D)$, and the set U of the lemma coinc—

coincides with the set discussed in the theorem. Now taking $l(e) = \int_D f(x)e(x)\,dx,\ e(x) \in L_p(D)$, we obtain the required assertion. Let us note that for $n = 1,\ r = 1$, system (3) reduces to the single equation $d^m u/dz^m = 0$, the set U becomes the collection of polynomials in z of degree not exceeding $m - 1$, and Theorem 1 gives the corresponding results for the approximation of functions of one variable z by polynomials, both in the metric $L_p(\Lambda)$ and in the Chebyshev metric on $\Lambda,\ \Lambda = \{\alpha \le z \le \beta\}$.

Now let system (3) be such that the set of forms $\{P_1(y), \dots, \dots, P_r(y)\}$ in the ring of polynomials in the variables $y_1, \dots, y_n$ forms a basis of a homogeneous ideal belonging to the set $\{a_1, \dots, a_k\}$.

Theorem 2. *If every section of the domain D by a hyperplane orthogonal to the direction a_i is connected, $i = 1, \dots, k$, then, in order that the generalized function $u = (u, \varphi(x)),\ \varphi(x) \in K_D$, be a sum of generalized plane waves of directions $a_1, \dots, a_k$, it is necessary and sufficient that the function u satisfy system (3).*

Moreover, if u is a solution of system (3) representable in the form

$$u = \sum_{|\gamma| \le m} \int_D f_{\gamma_1,\dots,\gamma_n}(x) \frac{\partial^{|\gamma|}\varphi(x)}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}}\,dx,$$

where $f_{\gamma_1,\dots,\gamma_n}(x) \in L_p^{\mathrm{loc}}(D)$, then there exist generalized functions h_i,

$$h_i = \sum_{j=1}^{m} \int_{\alpha_i}^{\beta_i} h_{ij}(z)\psi^{(j)}(z)\,dz,$$

where $h_{ij}(z_i) \in L_p^{\mathrm{loc}}(\Lambda_i),\ \psi(z_i) \in K_{\Lambda_i}$, such that

$$u = \sum_{i=1}^{k} h_i(a_i x),$$

and conversely.

The proof is carried out by induction on k. The following lemmas are used.

Lemma 4. *Let $\Delta_1' = \{x :\ \alpha_1' < x_n < \beta_1'\}$ be an arbitrary strip which, together with its boundary, lies inside $\Delta_1 = \{x :\ \alpha_1 < x_n < \beta_1\}$. Under the conditions of Theorem 2 there exist a curve $x_i = \tau_i(x_n),\ i = 1, \dots, n-1$, where τ_i is a polynomial, and a number $\varepsilon > 0$ such that the set of points x for which*

$$\tau_i(x_n) \le x_i \le \tau_i(x_n) + \varepsilon, \quad i = 1, \dots, n-1, \quad \alpha_1' \le x_n \le \beta_1',$$

belongs to the domain D.

Lemma 5. *Every function $\varphi(x),\ \varphi(x) \in K_{\Delta_1}$, can be written in the form*

$$\varphi(x) = \varkappa_1(x_1 - \tau_1(x_n)) \cdots \varkappa_{n-1}(x_{n-1} - \tau_{n-1}(x_n)) \int_{-\infty}^{\infty} \varphi(x)\, dx_1 \cdots dx_{n-1} + F(x),$$

where

$$F(x) = \sum_{i=1}^{n-1} \frac{\partial F_i(x)}{\partial x_i}, \qquad F_i(x) \in K_{\Delta_1'}, \qquad \varkappa_i(z) \in K_{(0,\varepsilon)},$$

$$\int_0^{\varepsilon} \varkappa_i(z)\, dz = 1, \qquad i = 1, \dots, n-1.$$

From Theorems 1 and 2 we obtain:

Theorem 3. *Let $f(x) \in L_p(D)$. If every section of the domain D by a hyperplane orthogonal to the vector a_i is connected, $i = 1, \dots, k$, then*

$$\inf \left\| f(x) - \sum_{i=1}^{k} h_i(a_i x) \right\|_p = \sup_{\|\tilde{\varphi}\|_q = 1} \left| \int_D f(x) \tilde{\varphi}(x)\, dx \right|, \tag{5}$$

where

$$\widetilde{\varphi} = \sum_{i=1}^{r} P_i \varphi_i(\mathbf{x}), \qquad \varphi_i(\mathbf{x}) \in K_D, \qquad q = \frac{p}{p-1},$$

and the lower bound is taken over all $h_i(z_i) \in H_i$,

$$\sum_{i=1}^{k} h_i(\mathbf{a}_i \mathbf{x}) \in L_p(D).$$

In this case there exist functions $h_i^0(z_i) \in H_i$ that realize the lower bound in (5).

Each of the sets $\Omega_1 = \overline{D} \cap \{\mathbf{x} : \mathbf{ax} = \alpha\}$ and $\Omega_2 = \overline{D} \cap \{\mathbf{x} : \mathbf{ax} = \beta\}$, where $\alpha = \inf_{\mathbf{x} \in D}(\mathbf{ax})$, $\beta = \sup_{\mathbf{x} \in D}(\mathbf{ax})$, $\mathbf{a} \in \Pi_{n-1}$, will be called a **supporting set** of the domain D for the direction $\mathbf{a}$. Any of the supporting sets Ω_{ij}, $j = 1, 2$, of the domain D, satisfying the conditions of Theorem 3, for each direction $\mathbf{a}_i$, $i = 1, \dots, k$, is a continuum. For fixed directions $\mathbf{a}_i$, $i = 1, \dots, k$, a supporting set Ω_{ij} of the domain D will be called **proper** in the following cases: 1) $\Omega_{ij} \cap \Omega_{st} = 0$, $s = 1, \dots, k$; $t = 1, 2$; $s \neq i$; 2) $\Omega_{ij} \cap \Omega_{s_0 t_0} \neq 0$, $s_0 \in \{1, \dots, k\}$, $t_0 \in \{1, 2\}$, $s_0 \neq i$, but the set Ω_{ij} contains the base of a half-sphere whose interior belongs to D.

Theorem 4. *If the requirements of Theorem 3 are satisfied and, among the directions $\mathbf{a}_i$, $i = 1, \dots, k$, it is possible to choose $k - 1$ directions so that each supporting set Ω_{ij} for every chosen direction is proper, then equality (5) holds under the condition that the lower bound is taken over all $h_i(z_i)$, $h_i(\mathbf{a}_i \mathbf{x}) \in$*

$L_p(D)$. Moreover, there exist functions $h_i^0(z_i)$, $h_i^0(\mathbf{a}_i\mathbf{x}) \in L_p(D)$, that realize the lower bound.

We give an example of a domain D and a system $\mathbf{a}_1, \dots, \mathbf{a}_k$ not satisfying the conditions of Theorem 4, for which there exists a function $f(\mathbf{x})$, $f(\mathbf{x}) \in L_p(D)$ for some p, $1 < p < \infty$, such that for it the lower bound (1) is not attained among functions $h_i(z_i)$, $h_i(\mathbf{a}_i\mathbf{x}) \in L_p(D)$.

Let

$$S = \sum_{\nu=1}^{\infty} \frac{1}{\nu^{1+\delta}},$$

where δ is a fixed number, $0 < \delta < 1/2$;

$$S_m = \sum_{\nu=1}^{m} \frac{1}{\nu^{1+\delta}}.$$

In the plane X_1OX_2 take the domain D bounded by the axis OX_1, the line $x_2 = x_1$, and the polygonal line with vertices (S_m, S_{m-1}), $m = 1, \dots$, $S_0 = 0$. Put $k = 2$, $\mathbf{a}_1 = (1,0)$, $\mathbf{a}_2 = (0,1)$. The chosen directions do not satisfy the conditions of Theorem 4. For $1 < p < 1+\delta$, for the function $f(x_1, x_2) = h_1(x_1) + h_2(x_2)$, where $h_1(x_1) = m^{1+2\delta}$, $S_{m-1} \leq x < S_m$; $h_2(x_2) = -m^{1+2\delta}(1 - 1/m^{2\delta})$, $S_{m-1} \leq x < S_m$, $m = 1, \dots$, the lower bound (1) is not attained.

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References

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Note: Figure translations are in progress. See original paper for figures.