SOME QUESTIONS OF ACKERMANN' S CALCULUS

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SOME QUESTIONS OF ACKERMANN'S CALCULUS

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The article considers the calculus of "strict implication"introduced by Ackermann ([1]). His aim was to obtain an implication reflecting, as far as possible, the properties of the connective "if ···, then." As is known, in classical logic the so-called paradoxes of material implication are true: "from falsehood anything follows," "truth follows from any statement" ; the truth of material implication does not presuppose a semantic connection between the antecedent and the consequent.

In Ackermann' s calculus Π', the paradoxes of material implication already turn out to be underivable. The same is true for E, a modification of the calculus Π' proposed by Anderson and Belnap ([2]).

The decision problem for the calculi under consideration has not yet been solved. In the present article some necessary conditions are established for the provability of formulas in SI (SI is the common designation for the formal systems Π' and E).

As was shown independently by Belnap ([4]) and Donchenko ([5]), the formula $\mathfrak{A} \to \mathfrak{B}$ is underivable in SI if $\mathfrak{A}$ and $\mathfrak{B}$ have no letters in common. For derivability, an even stronger dependence between the formulas $\mathfrak{A}$ and $\mathfrak{B}$ is needed.

Theorem 1. *If $\vdash \mathfrak{A} \to \mathfrak{B}$, then there is a variable which occurs positively in $\mathfrak{A}$ and $\mathfrak{B}$, or negatively in both formulas.*

Positive and negative occurrences of subformulas are defined here in the usual way: $\mathfrak{A}$ is a positive subformula of $\mathfrak{A}$; if $\neg\mathfrak{B}$ occurs positively (negatively) in $\mathfrak{A}$, then $\mathfrak{B}$ occurs in $\mathfrak{A}$ negatively (positively); if $\mathfrak{B}\&\mathfrak{C}$ or $\mathfrak{B} \vee \mathfrak{C}$ occurs positively (negatively) in $\mathfrak{A}$, then $\mathfrak{B}$ and $\mathfrak{C}$ both occur positively (negatively); if $\mathfrak{B} \to \mathfrak{C}$ occurs positively (negatively), then $\mathfrak{B}$ occurs negatively (positively) and $\mathfrak{C}$ occurs positively (negatively).

Theorem 1 is proved with the aid of the Ackermann matrix ([5]) given in ([4]).

Theorem 2. *Let the formula* $\mathfrak{A}$ *contain no positive subformulas of the form* $\mathfrak{B} \vee \mathfrak{C}$ *and no negative ones of the form* $\mathfrak{B} \& \mathfrak{C}$. *If* $\vdash \mathfrak{A}$, *then every variable occurring in* $\mathfrak{A}$ *occurs at least once positively and at least once negatively.*

Indeed, if the formula $\mathfrak{A}$ satisfies the conditions of the theorem and some variable A occurs in $\mathfrak{A}$ only positively (negatively), then $\mathfrak{A}$ cannot be proved in SI, since it is refuted by the following Ackermann matrix: the base set is $\{-2, -1, +1, +2\}$, the designated values are $\{+1, +2\}$,

$$\bar{x} = -x,$$

$$x \& y = \min(x, y),$$

$$x \vee y = \max(x, y);$$

$$x \to y = \begin{cases} \bar{x} \vee y, & \text{if } x \leqslant y, \\ \bar{x} \& y, & \text{if } x > y. \end{cases}$$

It is enough to assign to the variable A the value -2 (respectively $+2$), and to all the remaining variables the value $+1$. Then the formula $\mathfrak{A}$ will receive the undesignated value -2.

Thus, all occurrences of variables turn out to be, in a certain sense, connected; in particular, there can be no variable that occurs in the formula to be proved of the indicated form only once.

For the implicative fragment of the systems SI the theorem was proved by Anderson and Belnap ([3]).

Ackermann ([1]) proved the theorem: the formula $\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$ is not provable if $\mathfrak{A}$ does not contain implication. This theorem can be strengthened in the following direction.

Let $\mathfrak{M}$ be the set of formulas in which a positive occurrence of implication occurs only under the sign of another implication.

We define the class of admissible formulas as the class containing all axioms, formulas of the form $\mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{A}, \mathfrak{B}$ belong to $\mathfrak{M}$, and closed under the operations of inference $(\alpha), (\beta), (\delta)$ ([1]).

Theorem 3. *If* $\mathfrak{A}_1, \dots, \mathfrak{A}_k$ $(k \geq 0)$ *are admissible formulas,* $\mathfrak{A} \in \mathfrak{M}$, *then the formula*

$$\mathfrak{A}_1 \to (\mathfrak{A}_2 \to \cdots (\mathfrak{A}_k \to (\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C}))) \cdots)$$

is not provable in SI.

For the proof, consider the Ackermann matrix with basic set $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$ and designated elements $\{+1, +2, +3, +4, +5\}$.

Table 1

$x \to y$

$x\backslash y$	-5	-4	-3	-2	-1	$+1$	$+2$	$+3$	$+4$	$+5$
-5	$+2$	$+2$	$+2$	$+2$	$+2$	$+2$	$+2$	$+2$	$+2$	$+2$
-4	-5	$+2$	$+2$	$+2$	$+2$	-5	$+2$	$+2$	$+2$	$+2$
-3	-5	-5	$+2$	$+2$	$+2$	-5	-5	$+2$	$+2$	$+2$
-2	-5	-5	-5	$+2$	$+2$	-5	-5	-5	$+2$	$+2$
-1	-5	-5	-5	-5	$+1$	-5	-5	-5	-5	$+2$
$+1$	-5	-4	-4	-4	-4	$+1$	$+2$	$+2$	$+2$	$+2$
$+2$	-5	-4	-4	-4	-4	-5	$+2$	$+2$	$+2$	$+2$
$+3$	-5	-5	-4	-4	-4	-5	-5	$+2$	$+2$	$+2$
$+4$	-5	-5	-5	-4	-4	-5	-5	-5	$+2$	$+2$
$+5$	-5	-5	-5	-5	-5	-5	-5	-5	-5	$+2$

The implication $x \to y$ is given by Table 1, and the remaining operations as follows:

$$\bar{x} = -x,$$

$$x \& y = \max z \, (z \to x > 0 \text{ and } z \to y > 0),$$

$$x \vee y = \min z \, (x \to z > 0 \text{ and } y \to z > 0).$$

If all propositional variables take the value $+3$, then formulas belonging to $\mathfrak{M}$ take values from the set $\{-1, -2, -3, +3, +4, +5\}$; admissible formulas take values from the set of ···

the set $\{-1, -2, -3, -4, +2, +3, +4, +5\}$; the formula
$\mathfrak{A}_1 \to (\mathfrak{A}_2 \to \ldots (\mathfrak{A}_k \to (\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C}))) \ldots)$ takes the value -5 and, consequently, is not derivable in SI.

It is not hard to see that all formulas provable in SI are also derivable in the classical propositional calculus, but the converse is not true. However, it is possible to single out a class of formulas for which derivability in SI is equivalent to classical provability.

Theorem 4. If a formula $\mathfrak{A}$ contains no positive subformulas of the form $\mathfrak{B} \to \mathfrak{C}$, then $\mathfrak{A}$ is derivable in SI if and only if it is derivable in the classical calculus.

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Note: Figure translations are in progress. See original paper for figures.