# Integrability conditions of certain second order nonlinear differential equations

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## Abstract

The paper specifies the integrability criteria for three nonlinear second-order differential equations. Bibliography: 8 items.

## Full Text

## Introduction

In 1967, S. Ya. Bartashevich [1] investigated certain classes of nonlinear differential equations. Building upon these results, we consider the following general forms of second-order differential equations:

$$y'' + my' + f(x)y + g(x)y^n = e(x) \tag{1}$$

$$y'' + (m-1)y' + f(x)yy' + g(x)y^n = e(x) \tag{2}$$

$$[yy' + (m-1)y^2 + f(x)yy'] + g(x)y^n = e(x) \tag{3}$$

Equations of this type frequently appear in various physical and technical applications [2-6]. Following the methodology in [1], we apply the transformations $x = \alpha(t)$ and $y = \beta(t)z(t)$ to reduce equations (1)-(3) to autonomous or simpler forms, such as:

$$z'' + F(t)z' + \Phi(t)z + \Psi(t)z^n = 0 \tag{4}$$

$$z'' + mz' + F(t)z' + \Phi(t)z^n = \Psi(t) \tag{5}$$

$$[zz' + (m-1)z^2 + F(t)zz'] + F(t)z^n = \Phi(t) \tag{6}$$

By analyzing the coefficients $\Phi(t)$ and $\Psi(t)$, we can establish the integrability conditions for equations (1)-(3). For instance, the relationship between the coefficients of equation (1) and its transformed counterpart (5) can be expressed through the auxiliary functions $p$ and $q$, where $p = 1$ and $q = m$.

### Transformation and Integrability Conditions

For equation (2), assuming the transformation parameters $\alpha$ and $\beta$ satisfy specific constraints, we can derive the functional form of $f(x)$ that allows for exact solutions. When $e(x) = 0$, the relationship between the coefficients $f(x)$ and $g(x)$ is governed by the following differential relations:

$$u = \frac{f'(x)}{f(x)}; \quad \frac{F'(t)}{F(t)} \tag{13}$$

$$\frac{g(x)}{[f(x)]^{n+3}} = \frac{\Phi(t)}{[F(t)]^{n+3}} \tag{15}$$

Substituting these into the original equations, we obtain a system of conditions that define the class of integrable functions $f(x)$ and $g(x)$. Specifically, for equation (3), the transformation leads to:

$$\int f(x)dx = \int F(t)dt \tag{16}$$

The general solution can then be expressed in terms of the transformed variable $z(t)$, which satisfies a simpler autonomous equation.

### Special Cases and Applications

We further examine the case where $F(t) = a$ and $\Psi(t) = b$ are constants. As noted in [8], if we introduce the substitution $z' = \xi$, the problem reduces to a first-order equation:

$$\xi' + m\xi^2 + F(t)\xi + F(t) = 0 \tag{23}$$

Under the condition $4ma - b^2 = 0$, the solution simplifies significantly. Using the relations (13), (20), and (21), we can determine the explicit forms of $f(x)$ and $g(x)$ that satisfy the integrability criteria. The resulting general solution for $y(x)$ is then given by:

$$z = C_2(t - C_1)e^{-bt/2m} \tag{28}$$

where the relationship between $x$ and $t$ is defined by the integral of $f(x)$.

These results extend the known classes of integrable second-order nonlinear equations and provide a systematic framework for solving boundary value problems in mathematical physics where such structures arise.

## References

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*Note: Figure translations are in progress. See original paper for figures.*