EQUATIONS OF ELECTRODYNAMICS OF ANISOTROPIC MEDIA

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Abstract

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D. N. CHETAEV

EQUATIONS OF ELECTRODYNAMICS OF ANISOTROPIC MEDIA

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The article gives a complete system of equations for general electromagnetic potentials in homogeneous anisotropic media with an arbitrary tensor of electric permittivity.

1. We shall consider Maxwell' s equations for complex amplitudes of fields depending on time according to the law $\exp i\omega t$:

$$\operatorname{rot}\mathbf{E} = -i\omega\mu_0\mathbf{H}, \qquad \operatorname{rot}\mathbf{H} = i\omega\varepsilon_0\hat{\varepsilon}\mathbf{E} \tag{1}$$

with an arbitrary tensor of complex relative permittivity

$$\hat{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}. \tag{2}$$

For an effective solution of this system it is necessary to obtain equations for the field components separately. To this end, since system (1) in Cartesian coordinates x_1, x_2, x_3 is a linear system with constant coefficients, one can use the general method of eliminating unknowns ([1]). The algebraic elimination (according to Cramer' s rule), by virtue of the homogeneity of the system, gives a single homogeneous partial differential equation of order no higher than the sixth for any field component

$$LX = 0 \qquad (X = E_1, E_2, E_3, H_1, H_2, H_3), \tag{3}$$

where the linear operator L is the determinant composed of the operator coefficients of the system.

The form of the operator L is most easily obtained by first separating the equations for one of the fields, for example

$$\operatorname{grad}\operatorname{div}\mathbf{E} - \Delta\mathbf{E} - k_0^2\hat{\varepsilon}\mathbf{E} = 0 \qquad (k_0 = \omega/c). \tag{4}$$

The determinant of this linear system with constant coefficients,

$$\mathrm{Det} = \left| \left(\frac{\partial^2}{\partial x_i\,\partial x_k} - k_0^2\varepsilon_{ik} \right) - \delta_{ik}\Delta \right|, \tag{5}$$

where δ_{ik} is the Kronecker symbol, has the form of a secular determinant. Expanding it in powers of Δ, we obtain

$$\mathrm{Det} = -\Delta^3 + S\Delta^2 - C\Delta + D. \tag{6}$$

Here S is the analogue of the trace for the matrix (5), equal to

$$S = \Delta - k_0^2 \sum_i \varepsilon_{ii};$$

C is the sum of the principal minors of the coefficient matrix

$$C = -k_0^2\Delta \sum_i \varepsilon_{ii} + k_0^2 \sum_{i,k} \varepsilon_{ik} \frac{\partial^2}{\partial x_i\,\partial x_k} + k_0^4 \sum_i E_{ii},$$

where E_{ik} are the algebraic complements of the elements ε_{ik} of matrix (2).

The determinant of the coefficients of the characteristic matrix is equal to

$$D = k_0^4 \sum_{i,k} E_{ik} \frac{\partial^2}{\partial x_i\,\partial x_k} - k_0^6|\varepsilon_{ik}|,$$

where $|\varepsilon_{ik}|$ is the determinant of matrix (2). Substituting the expressions obtained into expansion (6), we find the operator of equation (3)

$$L \equiv \Delta \sum_{i,k} \varepsilon_{ik} \frac{\partial^2}{\partial x_i\,\partial x_k} + k_0^2\Delta \sum_i E_{ii} - k_0^2 \sum_{i,k} E_{ik} \frac{\partial^2}{\partial x_i\,\partial x_k} + k_0^4|\varepsilon_{ik}|, \tag{7}$$

which in the general case turns out to be an operator of fourth order. This fact could have been anticipated by considering system (1) and expanding the determinant formed from its operator coefficients by Laplace' s theorem with respect to the elements of the three rows corresponding to one of equations (1). Two minors of third order,

$$\begin{vmatrix} 0 & -\partial/\partial x_3 & \partial/\partial x_2 \\ \partial/\partial x_3 & 0 & -\partial/\partial x_1 \\ -\partial/\partial x_2 & \partial/\partial x_1 & 0 \end{vmatrix},$$

whose product could lead to a sixth-order operator, vanish identically as skew-symmetric.

2. In solving boundary-value problems of electrodynamics in terms of fields (or in terms of vector potentials with zero divergence, differing from the fields only by a constant factor in the absence of scalar potentials), the familiar difficulties may arise in satisfying the boundary conditions; these difficulties are the reason for introducing general electromagnetic potentials, also for isotropic media, which determine the field, for example, by the formulas

$$\mathbf{B} = \mu_0 \mathbf{H} = \operatorname{rot} \mathbf{A}, \qquad \mathbf{E} = -i\omega \mathbf{A} - \operatorname{grad} \Phi. \tag{8}$$

The introduction of potentials of the second kind leads to the same basic results; therefore we shall confine ourselves to considering $\mathbf{A}$ and Φ, which in an anisotropic medium satisfy the system of equations

$$\Delta \mathbf{A} - \operatorname{grad} \operatorname{div} \mathbf{A} + k_0^2 \hat{\varepsilon} \mathbf{A} - \frac{ik_0}{c} \hat{\varepsilon} \operatorname{grad} \Phi = 0,$$

$$\operatorname{div} \hat{\varepsilon} \operatorname{grad} \Phi + i\omega \operatorname{div} \hat{\varepsilon} \mathbf{A} = 0. \tag{9}$$

Investigating this linear system, which in Cartesian coordinates has constant coefficients, by the method of algebraic elimination of the unknowns, we find that the determinant of the matrix of its operator coefficients is identically zero. The equations of the system turn out to be linearly dependent, so that, for example, the last of them may be discarded. From the remaining three equations there follows a dependence among the four unknown functions, which may conventionally be expressed through the minors of the matrix of operator coefficients of system (9) as follows:

$$A_1 : A_2 : A_3 : \Phi = L\,\partial/\partial x_1 : L\,\partial/\partial x_2 : L\,\partial/\partial x_3 : -i\omega L. \tag{10}$$

The meaning of this notation is that

$$-i\omega L A_i = L \frac{\partial}{\partial x_i} \Phi, \tag{11}$$

whence, taking formulas (8) into account, we find that the potentials generating a field different from zero satisfy the equation

$$LX = 0 \qquad (X = A_1, A_2, A_3, \Phi). \tag{12}$$

On the other hand, conditions (11) allow the existence of potentials satisfying system (9), but not satisfying the last equa—

if only they are connected by the condition

$$\mathbf{A}^{(2)} = -\frac{1}{i\omega}\,\mathrm{grad}\,\Phi^{(2)}, \tag{13}$$

by virtue of which the field determined by them is equal to zero. At the same time no restrictions are imposed on the function $\Phi^{(2)}$.

3. The components of the potentials are not independent solutions of equation (12), being connected by a system. However, a system of three equations with four unknowns is indeterminate, which leads to the necessity of eliminating the scalar potential by means of some normalization condition on the potentials [(2)], which must ensure automatic fulfillment of equation (12) for Φ, if it is satisfied by the components $\mathbf{A}$.

Such a condition may, for example, be chosen in the form of the following generalized Lorentz condition [(3)]:

$$\Phi = -\frac{c}{ik_0}\,\mathrm{div}\,\hat{\varepsilon}\mathbf{A}, \tag{14}$$

where $\hat{\varepsilon}$ is some constant tensor $\{\varepsilon_{ik}\}$.

If condition (14) is satisfied by two pairs of potentials that determine one and the same field and are connected by the gradient transformation

$$\mathbf{A}' = \mathbf{A} - \frac{1}{i\omega}\,\mathrm{grad}\,\Psi, \qquad \Phi' = \Phi + \Psi, \tag{15}$$

then it is easy to see [(3)], substituting (15) into (14), that the function Ψ must satisfy the equation

$$L_2\Psi \equiv \left(\mathrm{div}\,\hat{\varepsilon}\,\mathrm{grad} + k_0^2\right)\Psi = 0. \tag{16}$$

This equation may also be obtained from consideration of the determinant of the matrix of operator coefficients of the system

$$\Delta\mathbf{A} - \mathrm{grad}\,\mathrm{div}\,\mathbf{A} + \hat{\varepsilon}\,\mathrm{grad}\,\mathrm{div}\,\hat{\varepsilon}\mathbf{A} + k_0^2\hat{\varepsilon}\mathbf{A} = 0, \tag{17}$$

obtained as a result of eliminating Φ by means of condition (14) from the three equations (9). As direct calculation shows, this determinant is equal to the product of the operators L and L_2.

Thus, the general representation of the potentials in an anisotropic medium under the imposed generalized Lorentz condition has the form

$$\mathbf{A} = \mathbf{A}^{(1)} + \mathbf{A}^{(2)}, \tag{18}$$

where the components $\mathbf{A}^{(1)}$ satisfy the basic equation (12), while the components $\mathbf{A}^{(2)}$ satisfy equation (16), which from the standpoint of determining the field is redundant.

4. However, the introduction into consideration of the solutions of this equation makes it possible to determine uniquely the vector potential, under the chosen generalized Lorentz condition, satisfying the boundary conditions for the potentials.

Just as the components $\mathbf{A}^{(2)}$ are not independent solutions of equation (16), but are expressed, by virtue of condition (13), in terms of solutions for Ψ, the components $\mathbf{A}^{(1)}$ are connected by the system (17) and are expressed in terms of the solutions for one of them by means of the minors of the matrix of its operator coefficients. Thus we may have at our disposal solutions of equation (12) for only one function. This equation is not of sixth order, but of fourth order, and in the number of independent solutions is equivalent to two second-order equations in accordance with the two types of waves that can propagate independently in anisotropic media. As the case of an isotropic medium shows, in the presence of all components of the potentials, in order to be able to satisfy the boundary conditions we must have independent solutions of three second-order equations.

The use of solutions of the redundant equation also makes it possible, in the case of anisotropic media, to satisfy the boundary conditions in a regular manner and to find the unique values of the potentials under the chosen normalization condition.

The form of the superfluous equation depends on the tensor of the generalized Lorentz condition; however, its solutions are excluded in determining the field and do not correspond to any real types of waves. On the other hand, the choice of this tensor affects system (17), which determines the relations between the components, and in some cases may prove optimal in a certain sense, as was shown [4] in the case of a uniaxial medium for the normalization condition of A. N. Tikhonov [2], who first proposed using vector potentials with a nonzero scalar potential for solving boundary-value problems in anisotropic media.

Institute of Physics of the Earth
named after O. Yu. Schmidt
Academy of Sciences of the USSR

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[4] D. N. Chetaev, *Physics of the Earth*, No. 10 (1966).

Note: Figure translations are in progress. See original paper for figures.