GENERAL INTEGRAL REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS

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GENERAL INTEGRAL REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS

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In the present note, in the case of several complex variables, a general integral formula is established for functions holomorphic in convex complete n-circular domains of the space C^n, $n \geq 2$ (Theorem 4), and a general integral formula for functions holomorphic in convex domains of the space C^n, $n \geq 2$* (Theorem 5). Each of these two formulas, for every value of k from $\{1, 2, \dots, \mu\}$, includes an infinite set of integral representations. In formulating Theorems 4 and 5 we adhere to the definitions and notation used in (^{1}). One of the auxiliary formulas is the general integral formula obtained here in the case of one complex variable (Theorem 3) for functions holomorphic in star-shaped domains.

1. Let G be a star-shaped** domain with respect to the origin in the space C^n of complex variables $z_1, \dots, z_n$, $n \geq 1$, and let the function $F(z_1, \dots, z_n)$ be holomorphic in G. Further, let γ be an arbitrary positive number; p, q natural numbers with $p \geq q$, and let $\gamma_p, \gamma_{p-1}, \dots, \gamma_q, \gamma_0$ be arbitrary positive numbers. Introduce the notation (for brevity, instead of $F(z_1, \dots, z_n)$ we write F)

$$L_\gamma[F] = \gamma F + \sum_{\nu=1}^{n} z_\nu F'_{z_\nu}, \qquad L_{q,p}^{(p-q+1)}[F] = L_p[L_{p-1} \dots [L_q[F]] \dots],$$

$$L^{(p-q+1)}_{\binom{\gamma_q}{\gamma_p}}[F] = L_{\gamma_p}[L_{\gamma_{p-1}} \dots [L_{\gamma_q}[F]] \dots]$$

and put

$$L^{(0)}_{p,p-1}[F] = F, \qquad L^{(0)}_{\binom{\gamma_p}{\gamma_{p-1}}}[F] = F \quad (p \geq 1).$$

Theorem 1. *If the function $F(z_1, \dots, z_n)$ $(n \geq 1)$ is holomorphic in the domain G, then for every natural k in the domain G we have*

$$F(z_1, \dots, z_n) = \int_0^1 d\varepsilon_1 \dots \int_0^1 d\varepsilon_{k-1} \int_0^1 \varepsilon_1^{\gamma_1 - 1} \dots \varepsilon_k^{\gamma_k - 1} \times$$

$$\times L^{(k)}_{\binom{\gamma_1}{\gamma_k}} [F(\varepsilon_1 \dots \varepsilon_k z_1, \dots, \varepsilon_1 \dots \varepsilon_k z_n)] \, d\varepsilon_k, \tag{1}$$

where $\varepsilon_1, \dots, \varepsilon_k$ are real and $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers satisfying the condition $\gamma_j \geq 1$ $(j = 1, \dots, k)$.

For $k = 1$ formula (1) is proved in the same way as formula (1) in the author' s note (^{1}), and then for any natural k by induction.

2. Let G_1 be a star-shaped domain with respect to the origin in the space C^1, bounded by a closed rectifiable Jordan curve Γ, and $F(z)$ is a function holomorphic in the domain G_1. On the basis of formula (1) $(n = 1)$ and Cauchy' s formula $(n = 1)$ one obtains

Theorem 2. *If the function $F(z)$, holomorphic in the domain G_1, and all its derivatives up to order μ $(\mu > 0)$ inclusive are continuous in the closed domain $\overline{G}_1$, then for $k = 1, \dots, \mu$ and $z \in G_1$ the formula holds*

$$F(z) = \frac{1}{2\pi i} \int_\Gamma K \begin{pmatrix} \gamma_1 \\ \gamma_k \end{pmatrix} (\xi, z) \, L^{(k)} \begin{pmatrix} \gamma_1 \\ \gamma_k \end{pmatrix} [F(\xi)] \, d\xi, \tag{2}$$

where

$$K \begin{pmatrix} \gamma_1 \\ \gamma_k \end{pmatrix} (\xi, z) = \int_0^1 d\varepsilon_1 \dots \int_0^1 d\varepsilon_{k-1} \int_0^1 \frac{\varepsilon_1^{\gamma_1 - 1} \dots \varepsilon_k^{\gamma_k - 1}}{\xi - \varepsilon_1 \dots \varepsilon_k z} \, d\varepsilon_k, \tag{3}$$

$\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers satisfying $\gamma_j \geq 1$ $(j = 1, \dots, k)$, and integration is performed over the contour Γ in the positive direction.

Let us point out the distinctive features of the integral representation (2): a) it expresses the values of the function $F(z)$ in the domain G_1 through the values of the operator

$$L^{(k)} \begin{pmatrix} \gamma_1 \\ \gamma_k \end{pmatrix} [F]$$

on the boundary of the domain G_1; b) since $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers satisfying $\gamma_j \geq 1$ $(j = 1, \dots, k)$, formula (2), for each value of k from $\{1, 2, \dots, \mu\}$, includes an infinite set of integral representations.

Putting

$$K \begin{pmatrix} \gamma_1 \\ \gamma_0 \end{pmatrix} (\xi, z) = \frac{1}{\xi - z},$$

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* The same is true in the case of star-shaped domains in C^n, $n \geq 2$.

** A domain star-shaped with respect to the origin is called a domain containing, together with each point, the entire segment connecting this point with the origin.

one can, in the case of the domain G_1, combine Cauchy' s formula and formula (2) into a single formula (2). Thus we obtain the following general theorem 3.

Theorem 3. *If the function $F(z)$, holomorphic in the domain G_1, and all its derivatives up to order μ ($\mu \geq 0$; $F^{(0)} = F$) inclusive are continuous in the closed domain $\overline{G}_1$, then for $k = 0, 1, \dots, \mu$ and $z \in G_1$ formula (2) holds, where $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers satisfying $\gamma_j \geq 1$ ($j = 1, \dots, k$).*

3. Let us pass to the case $n \geq 2$. Let B and H be bounded star-shaped domains with respect to the origin in C^n, respectively with piecewise-smooth and smooth boundaries.

Theorem 4. *Let $D \in (T)$, and let the function $f(z)$ ($n \geq 2$) be holomorphic in D, and let α be a number equal to 0 or 1. Then, if the functions*

$$f_{z_\nu^{(\alpha)}}(z), \qquad \nu = 1, \dots, n^{**},$$

and all their partial derivatives up to order μ ($\mu \geq 0$) inclusive are continuous in $D \cup S$, then for $k = 0, 1, \dots, \mu$ and $z \in D$

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[*] We note that if the domain G_1 is the disk $|z| < 1$, then under the conditions of theorem 3, for $k = 0, 1, \dots, \mu$ and $|z| < 1$ the formulas

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} P\begin{pmatrix} \gamma_1 \\ \gamma_k \end{pmatrix} (\rho, \varphi - \psi)\, L^{(k)} \begin{pmatrix} \gamma_1 \\ \gamma_k \end{pmatrix} [F(e^{i\varphi})]\, d\varphi \quad (z = \rho e^{i\psi}),$$

$$F(z) = i\ \mathrm{Im}\, F(0) + \frac{1}{2\pi} \int_0^{2\pi} S\begin{pmatrix} \gamma_1 \\ \gamma_k \end{pmatrix} (e^{i\varphi}, z)\ \mathrm{Re}\, L^{(k)} \begin{pmatrix} \gamma_1 \\ \gamma_k \end{pmatrix} [F(e^{i\varphi})]\, d\varphi,$$

hold, where the kernels

$$P\begin{pmatrix} \gamma_1 \\ \gamma_k \end{pmatrix} (\rho, \varphi - \psi) \quad \text{and} \quad S\begin{pmatrix} \gamma_1 \\ \gamma_k \end{pmatrix} (e^{i\varphi}, z)$$

for $k = 1, \dots, \mu$ have the form obtained from (3) by replacing in (3)

$$\frac{1}{\xi - \varepsilon_1 \dots \varepsilon_k z}$$

respectively by

$$\frac{1 - (\varepsilon_1 \dots \varepsilon_k \rho)^2}{1 + (\varepsilon_1 \dots \varepsilon_k \rho)^2 - 2\varepsilon_1 \dots \varepsilon_k \rho \cos(\varphi - \psi)}$$

and

$$\frac{e^{i\varphi} + \varepsilon_1 \dots \varepsilon_k z}{e^{i\varphi} - \varepsilon_1 \dots \varepsilon_k z},$$

and for $k = 0$ these kernels are, respectively, the Poisson and Schwarz kernels.

[**] If $\alpha = 0$, then these functions are one and the same function $f(z)$.

$$f(z) = \alpha f(0) + \frac{1}{n+\alpha(1-n)} \sum_{\nu=1}^{n} \frac{z_\nu^\alpha}{(2\pi)^{n_i}} \int d\omega_* \int d\omega_\theta \int_{|\xi|=1} L_{\alpha+1,n-1}^{(n-1-\alpha)} \left[K \begin{pmatrix} \gamma_1 \\ \gamma_k \end{pmatrix} (\xi, \mu) \right] \times$$

$$\times L \begin{pmatrix} \gamma_1 \\ \gamma_k \end{pmatrix}^{(k)} \left[F_{0 z_\nu^\alpha}^{(\alpha)} (\xi, r, \theta) \right] d\xi, \tag{4}$$

where $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers satisfying $\gamma_j \geq 1$ $(j = 1, \dots, k)$.*

Theorem 5. Let the function $F(z_1, \dots, z_n)$ $(n \geq 2)$ be holomorphic in the domain D^{**}, and let α be a number equal to 0 or 1. Then, if the functions $F_{z_\nu^\alpha}^{(\alpha)}(z_1, \dots, z_n)$, $\nu = 1, \dots, n$, and all their partial derivatives up to order μ $(\mu \geq 0)$ inclusive are continuous in the closed domain $\overline{D}$, then for $k = 0, 1, \dots, \mu$ and points $(z_1, \dots, z_n) \in D$ there holds a formula (in view of the brevity of the presentation we do not write it out here), analogous in character to formula (4), expressing the values of the function $F(z_1, \dots, z_n)$ in the domain D through the values

$$L \begin{pmatrix} \gamma_1 \\ \gamma_k \end{pmatrix}^{(k)} \left[F_{z_\nu^\alpha}^{(\alpha)} \right],$$

where $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers satisfying $\gamma_j \geq 1$ $(j = 1, \dots, k)$, on the boundary of the domain D, up to the summand $\alpha F(0, \dots, 0)$. Similarly in the case of the domains B and H^{***}.

For the proof of Theorem 4 one should use Theorem 3 of this note and the integral formula (for $k = 0$) in Theorem 2 from (^{1}); for the proof of Theorem 5 in the case of the domain D, formula (1) of the present note, formula (2) from (^{1}), and formula (2) from (^{2}); while in the case of the domains B and H the only difference is that instead of formula (2) from (^{2}) one uses, respectively, the Martinelli-Bochner integral representation (see, for example, (^{3})) and integral formula (2) from (^{4}).

Remark 1. Each of the four integral formulas in Theorems 4 and 5 (in Theorem 5 three formulas are meant: one in the case of the domain D, another for the domain B, and a third for H) for any value of k from $\{1, 2, \dots, \mu\}$ includes an infinite set of integral representations, since $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers satisfying $\gamma_j \geq 1$ $(j = 1, \dots, k)$.

4. Formula (1) $(k = 1)$ remains valid also in the case when γ_1 is any positive number. But for $0 < \gamma_1 < 1$ the integral entering this formula should be understood as an improper one. Taking into account the same remark concerning analogous integrals, Theorem 1 and all the other results of this note connected with $\gamma_1, \dots, \gamma_k$ remain valid also in the case when $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers.

5. With the aid of his integral representations, A. A. Temlyakov ([5,6]) obtained integral representations for a certain class of meromorphic functions in the case of two complex variables. In an analogous way, with the aid of Theorem 4 of the present note, integral representations are established for the corresponding class of meromorphic functions of n ($n \geq 2$) complex variables.

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* Under the conditions of Theorem 4 there are also two more general formulas corresponding to the formulas indicated in the footnote on p. 1252.

** D here is the same convex domain in C^n that was considered in ([1,2]).

*** The integral formula in the case of the domain H is more general in construction than in the case of the domain B.

Note: Figure translations are in progress. See original paper for figures.