Solution of problems in the theory of heat and mass transfer

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Abstract

The paper presents the application of the contour integral method to solving mixed problems for a system of differential equations of heat and mass transfer under molecular and molar transport of energy and matter in an arbitrary three-dimensional domain. Furthermore, the contour integral method is used to construct the fundamental solution matrix of the aforementioned system in closed form, a compact formula representing the solution to the Cauchy problem is provided, and a scheme for applying the thermal potential theory method to solving a mixed problem that does not contain a time derivative in the boundary condition is indicated. Bibliography: 2.

Full Text

Preamble

This work, published in 1967, addresses boundary value problems for a specific class of partial differential equations. We consider the vector $x = (x_1, x_2, x_3)$ and an operator A defined by coefficients a_{ij} $(i, j = 1, 2, 3)$. Following the methodology established in [**?**], we investigate the properties of the solution $v(x, t)$ within a domain D.

§ 1. Problem Statement and Fundamental Solutions

Consider the equation:

$$\Delta v - \lambda^2 v = \Phi(x)$$

subject to the boundary conditions on Γ:

$$B(z, \lambda)v = \psi(z), \quad z \in \Gamma$$

and the initial condition:

$$v(x, 0) = \Phi(x)$$

We assume the following conditions hold: 1. The operator (1) satisfies the ellipticity conditions defined by I. G. Petrovsky. 2. The coefficients $a_k(z)$ and $P_k(z)$ $(k = 0, 1)$ are continuous on Γ, and the determinant $\det(a_0(z)+\lambda a_1(z)) \neq 0$. 3. The function $\Phi(x)$ is defined and sufficiently smooth in the domain D.

The characteristic equation associated with (1) is given by:

$$\delta(\mu) = \det(I\mu^2 + \sum a_i \mu + b) = 0$$

where b_{ks} are coefficients related to the operator. Let v_1, v_2, v_3 be the roots of the characteristic polynomial:

$$v^3 + (a_{11} + a_{22} + a_{33})v^2 + (b_{11} + b_{22} + b_{33})v + \delta = 0$$

Under the assumption that the roots v_k are distinct, the fundamental solution $P(x, \lambda)$ can be expressed via the roots of the characteristic equation. Specifically, we define the components $P_{mn}(x, \lambda)$ as:

$$P_{mn}(x, \lambda) = \frac{1}{8\pi} \sum_{k=1}^{3} \frac{q_{mn}(v_k)}{\prod_{j \neq k}(v_k - v_j)} \frac{e^{-v_k|x|}}{|x|}$$

where $q_{mn}(v)$ are polynomials in v determined by the coefficients of the original system.

§ 2. Construction of the Green’ s Function

To solve the boundary value problem, we introduce the potential $u_1(x, \lambda)$ and the integral representation:

$$u_1(z, \lambda) + \int_\Gamma K(z, y, \lambda)\mu(y, \lambda)d\Gamma_y = \psi(z, \lambda)$$

where the kernel $K(z, y, \lambda)$ is defined using the fundamental solution $P(z-y, \lambda)$ and the boundary operator coefficients. For sufficiently large $|\lambda| > R$, the kernel satisfies the estimate:

$$|K(z, y, \lambda)| \leq \frac{C}{|\lambda|} \exp(-\epsilon|\lambda||z - y|)$$

This ensures the existence of a unique solution for the density $\mu(y, \lambda)$ via a Neumann series or Fredholm theory, depending on the domain geometry.

The Green’ s function $G(x, \xi, \lambda)$ for the problem is then constructed as:

$$G(x, \xi, \lambda) = P(x - \xi, \lambda) - Q(x, \xi, \lambda)$$

where $Q(x, \xi, \lambda)$ is the compensating term that ensures the boundary conditions are satisfied. We prove that for $x, \xi \in D$, the function Q and its derivatives satisfy exponential decay estimates proportional to $\exp(-\epsilon|\lambda||x - \xi|)$.

§ 3. Non-Stationary Solutions

The solution to the time-dependent problem (1)-(5) can be obtained using the inverse Laplace transform:

$$v(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\left[\int_D G(x,\xi,\lambda)\Phi(\xi)dD_\xi\right]d\lambda$$

By analyzing the asymptotic behavior of $G(x,\xi,\lambda)$ in the complex λ-plane, we establish the existence and uniqueness of the solution $v(x,t)$.

For the case where $\Phi(x) = 0$ and the boundary conditions are non-homogeneous, the solution is represented as:

$$v_2(x,t) = \int_0 t d\tau \int_\Gamma Q(x-y, t-\tau)\mu(y,\tau)d\Gamma_y$$

where the density $\mu(z,t)$ satisfies a Volterra integral equation of the second kind:

$$\mu(z,t) = 2a^{-1}(z,t)\psi(z,t) + \int_0^t d\tau \int_\Gamma R(z,y,t-\tau)\mu(y,\tau)d\Gamma_y$$

The kernel R is derived from the fundamental solution of the parabolic or hyperbolic system, ensuring the continuity of the solution up to the boundary.

References

1. Ladyzhenskaya, O. A., Solonnikov, V. A., Uraltseva, N. N. *Linear and Quasi-linear Equations of Parabolic Type.* Nauka, 1967.
2. Rikhl, M. L. *Mathematical Methods in Physics.* Nauka, 1964.

Note: Figure translations are in progress. See original paper for figures.