# The concept of stability with respect to a generalized perturbation

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## Abstract

In the article "Stability of Generalized Processes. I" ( "Differential Equations" , 1966, No. 7, pp. 872–881), the concept of stability with respect to generalized actions was introduced. The purpose of this article is to present those perspectives on classical stability that served as the starting point for defining the types of stable transformations of generalized perturbations performed by dynamical systems. Bibliography: 9 items.

## Full Text

## Preamble

This work, published in 1967 (Vol. III, No. 12), builds upon the foundational research of S. T. Zavalishchin [?] and other recent developments in the field [?, ?]. Specifically, we address the operator-based approaches discussed in [?] and the methodologies established in [?]. The study also incorporates insights from E. A. Barbashin regarding the stability of motion. Our primary objective is to refine the concept of "generalized solutions" in the context of differential equations with distributions, particularly focusing on the results presented by S. T. Zavalishchin in [?]. We aim to clarify the relationship between the spaces of solutions and the spaces of input perturbations.

### § 1. General Definitions and Stability

Consider the linear differential equation:

$$\dot{x} = A(t)x, \tag{1.1}$$

where $x$ is an $n$-dimensional vector and $A(t)$ is an $n \times n$ matrix. Let $U(t)$ be the fundamental matrix of the homogeneous system (1.1). We extend this to the non-homogeneous case involving a distribution $\eta$:

$$d\mu = A(t)\mu \, dt + d\eta, \tag{1.2}$$

The solution to (1.2) can be represented using the Cauchy formula:

$$\mu(t) = W(t,0)\mu_0 + \int_0^t W(t,s)\,d\eta(s), \tag{1.3}$$

where $W(t,s) = U(t)U^{-1}(s)$ is the transition matrix. For an initial condition $\mu_0 = 0$, the behavior of the solution on the interval $0 \le t < \infty$ is determined by the properties of the integral term. Following the conventions in [?, ?], we analyze the mapping from the space of perturbations $\eta$ to the space of solutions $\mu$.

Let $V(\infty)$ denote the space of functions of bounded variation on $[0,\infty)$. We define the operator $Q$ that maps a perturbation $\eta \in V(\infty)$ to a solution $\mu$ via the relation:

$$\mu = Q\eta = \int_0^t W(t,s)\,d\eta(s). \tag{1.4}$$

The stability of this mapping is characterized by the norm:

$$\|\eta\|_V = \sup_{t\ge 0} \|\eta(t)\| \tag{1.5}$$

and the corresponding norm for the solution space. We say the system is $(D_0, D)$-stable if the operator $Q$ maps the space $D_0$ into $D$ continuously. As shown in [?], a necessary condition for the boundedness of the operator $Q$ is:

$$\sup_{t\ge t_0} \int_{t_0}^t \|W(t,s)\|\,ds < \infty. \tag{1.7}$$

Under the assumption that the matrix $A(t)$ is bounded, i.e.,

$$\sup_{t\ge 0} \|A(t)\| < \infty, \tag{1.8}$$

the stability condition (1.7) is equivalent to the existence of constants $M, \alpha > 0$ such that:

$$\|W(t,s)\| \le Me^{-\alpha(t-s)} \quad (0 \le s \le t < \infty). \tag{1.9}$$

## § 2. Generalized Solutions and Distributions

In [?], Zavalishchin introduced the space of distributions $K_+$ and defined generalized solutions for systems where the input $\eta$ belongs to this class. Let $VK_+$ be the space of distributions whose primitives belong to $V(\infty)$. We consider the mapping $Q$ in the context of these generalized functions:

$$\mu = Q\eta, \quad \eta \in VK_+. \tag{2.1}$$

The relationship between the classical solution (1.4) and the generalized solution (2.2) is central to our analysis. If the system (1.1) is exponentially stable, then the operator $Q$ defined by (2.3) provides a continuous mapping between the respective distribution spaces.

Specifically, for any $\eta \in V(\infty)$, the solution $\mu$ defined by (1.4) satisfies:

$$\|\mu(t)\| \leq W_0 \cdot \mathrm{Var}(\eta), \tag{1.10}$$

where $W_0$ is a constant depending on the stability of $A(t)$. This confirms that the integral operator $Q$ is well-behaved under the conditions of exponential stability (1.11).

### Conclusion

The results presented here demonstrate that the framework of $(D_0, D)$-stability provides a robust mechanism for analyzing differential equations driven by distributions. By utilizing the transition matrix $W(t, s)$ and the operator $Q$, we can ensure that generalized solutions remain within predictable bounds, provided the underlying homogeneous system (1.1) satisfies standard stability criteria [**?**, **?**, **?**].

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## Figures

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# THE CONCEPT OF STABILITY WITH RESPECT TO GENERALIZED PERTURBATIONS

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**Introduction.** The occurrence of perturbations, which violate the desired character of the development of a real process, always proceeds in time. However, when determining the corresponding deviations of the process, the time of the onset of the actions is usually neglected [6]. As a result, when analyzing automatic control systems, idealizations such as the Heaviside function, impulse and other functions appear. Therefore, one has to face the following problems:

1) finding the reactions of the object to perturbations containing derivatives of some orders from discontinuous functions;

2) elucidating the conditions under which small input signals, having as components derivatives of some orders from discontinuous functions, are transformed into small output signals.

The not less difficult problem of defining the concept of "smallness" in relation to the processes under discussion is connected with the latter question. The attempt to solve all these questions by means of classical analysis is devoid of internel logic, and often [7] leads to incorrect results. The way out of this contradiction is to involve the apparatus of generalized functions, which allows one to obtain information that agrees with experimental data. Along this path, stσm in monographs [6—8], possible techniques for determining the reaction of the system to generalized influences berlonging to some special classes are discussed. In the book [8] and article [9], questions of the asymptotic behavior of the reaction of the sytem to a series of impulse perturbations are partially touched upon. However, the first fundamental investigation in this direction was carriyled out by E. A. Barbashin [1]. In the class of influences, which are generalized derivatives of functions of locally bounded variation, sudclassses were found which the system, possassing one or another type of intrinsic stability, is in a state to parry.

In this same article, the problem of stability is posed with respect to perturbations of a more complex nature — generalized processes. The main difficulty that arises when solving this question is connected with the problem of estimating the magnitude of a generalized function. The reason for such a complication is that, in contrast to classical processes, the value of a generalized process at individual moments of time, generally rebapinating, cannot be given any clear quantitative sense. Characterizing the situation as a whole, it is necossary to note that the more complex the structure of the generalized functions involved, the more abstract becomes the concept of a "small" generalized process. Thus, for hanpuσd, in article [4], where arbitrary generalized functions were taken as admissible influences, it was only possible to introduce the concept of an infiniαly small process as a sequence of generalized . . . divepearing according to some rule.

Figure 1: Figure 1

functions. At the same time, the study of the system's stability reduced to elucidating the continuity of the mapping of the space of generalized processes defined by it, equipped with one or another type of convergence. On this path in the note [4], the concept of stability with respect to generalized perturbations was introduced. However, having paid considerable attentantion to the mathematical side of the problem, the author omitted the presentation of perspectives on classical stability, which served as a starting point in determining the types of stabive transformations of generoalized perturbations, carried out by dynamic systems. The purpose of this note is to fill the gap that has formed, i.e., to elucidate the classical content of the concept of stability with respect to generalized perturbations introduced in [4].

## § 1. DISCUSSION OF CLASSICALLY STABLE SYSTEMS FROM THE POINT OF VIEW OF THE TOPOLOGICAL CHARACTER DEFINED BY THEM MAPPINGS OF THE SPACE OF FUNCTION OF LOCALLY FINITE VARIATION *)

Let a system of differential equations be given

$$\dot{\mu} = A(t)\,\mu, \tag{1.1}$$

where $\mu$ is an $n$-dimensional vector; $A(t)$ is an $n \times n$ matrix with variable eleements. We will assume that the matrix $A(t)$ is such, ꞉ the solution to the Cauchy problem is possible in the entire Euclidean space $E^n$ and is extendable for $t \geqslant 0$. Then the solution with the initial state $\mu_0$ can be represented in the form $\mu = U(t)\mu_0$, where $U(t)$ denotes the fundamental matrix of solutions of the first system (1.1). Along with the system (1.1), we will consider the perturbed system, the equations of which we write in the form of the following differential relation:

$$d\mu = A(t)\,dt\,\mu + d\eta. \tag{1.2}$$

In the system (1.2), the function $\eta$ describes the perturbing action. Let us assume that it belongs to the space $V(\infty)$. Then the solution to the Cauchy problem exists and is given by the Cauchy formula, which, utilizing the concept of the Stieltjes ıntegrapl, we write in the form

$$\mu(t) = W(t, 0)\,\mu_0 + \int_0^t W(t, s)\,d\eta(s). \tag{1.3}$$

In the relation (1.3) $W(t, s) = U(t)\,U^{-1}(s)$ is the Cauchy operator.

In the problem of stability to constantly aacting perturbations, herowsperturbations are pascmatred, that possess sucuficiently good differentrial differential cooproductries (for example, absolutely difeeerentiable), and it is assumed, that at the initial moment of time, the system is sot at the origin time, i.e., $\mu_0 = 0$.

Let us turn to stability according to Lyaponov [3]. In this clyse, we have $\eta(t) \equiv 0$ for $0 < t < \infty$ and in the formyle (1.3) the integral term vanishes. However, the first trem allows the expression [1]

$$W(t, 0)\,\mu_0 = \int_0^t W(t, s)\,d\chi(s)\,\mu_0, \quad t > 0,$$

*) As in the works [4, 5], in this note, we will denote the space $V(\infty)$ by $\eta \in V(\infty)$, then $\eta(t) = 0$ for $t \leqslant 0$.

Figure 2: Figure 2

where $\chi(s)$ is the Heaviside function (unit step). Hence, the stability of the zero solution of system (1.1) in the sense of Lyapunov can be interpreted as stability with respect to perturbations of the Heaviside function type. Thus, the problem of stability in the sense of Lyapunov is embedded [2] in the problem of stability under constantly acting perturbations, if the admissible class of the latter is expanded to the space $V(\infty)$. Assuming that $\eta$ is an arbitrary element of this space, for the response of system (1.2) to this action, we will use the expression

$$\mu = \int_0^t W(t,s)\, d\,\eta(s). \tag{1.4}$$

In work [1], relation (1.4) is interpreted as an operator mapping some functional space $D$ of elements from $V(\infty)$ into the space $D_0$ of functions from $V(\infty)$. In this case, certain types of asymptotic behavior of trajectories of the free system (1.1) are characterized in terms of spaces $D_0$, $D$. These results serve as source material in defining defining such an approach to the concept of stability, which allows formulating the concept of stability of generalized processes. Namely, we will consider the Cauchy formula (1.4) as a transformation of the space $V(\infty)$ into itself. With each of the types of spaces $D$ and $D_0$ considered in [1], we associate some special topology in the space $V(\infty)$. This correspondence can be established in such a way that the requirement that opentor (1.4) map some space $D$ into space $D_0$ turns out to be equivalent to the requirement of continuity of operator (1.4) when mapping space $V(\infty)$ (endowed with a topology corresponding to the type of space $D$) into space $V(\infty)$ with a top--ology corresponding to space $D_0$.
Thereby, the types of asymptotic behavior of trajectories of system (1.1) mentioned above will be characterized in terms of continuity of transformation (1.4) of space $V(\infty)$ into itself. On this path, the concept of stability is freed from the now non-rn-essential concept of state of the process and appears as a natural starting point of view on stability in an attempt to define the concept of stability of generalized processes.

Now, in accordance with the outlined plan, we proceed to discuss the main results [1] concerning the dependencies between the classical types of stability of system (1.1) and the types of functional spaces $D_0$, $D$. Preliminarily, it is necessary to determine a number of subspaces of the space $V(\infty)$. The set of functions from $V(\infty)$, for which

$$\|\eta\|_{\beta} = \sup_{t\to 0} \|\eta(t)\|_{en} < \infty, \tag{1.5}$$

we will denote by $\mathcal{B}$. The space of functions having finite variation on the semiaxis $[0,\infty)$, we will denote by $V(0,\infty)$. Those functions from the space $V(\infty)$, for which

$$\|\eta\|_{b} = \sup_{t\to t} \overset{t+1}{\underset{0}{\mathrm{V}}}\, \eta < \infty \quad (\overset{B}{\underset{\alpha}{V}}\eta - \text{vapuation } \eta), \tag{1.6}$$

will constitute the space $\nu$ [1]. Obviously, the inclusions $V(0,\infty) \subseteq \nu$, $V(0,\infty) \subseteq B$ hold. The spaces $B$ and $\nu$ intersect. For example, the function $\eta = t$ belongs to space $\nu$, but $\eta \notin B$. On the other hand, the function

Figure 3: Figure 3

be $D_0$-convergent. Therefore, by the definition of $D_0$-convergence, such a $k_0$ at $\Omega\eta_{k_0} \subseteq D_0$. But $\Omega\eta_{k_0} = \frac{1}{k_0}\Omega\eta$ is, thus, and, thus, $\Omega\eta$ also belongs to the space $D_0$, which contradicts the assumption. It remains to establish that the operator $\Omega$ is bounded. Since the operator $\Omega$ maps the space $D_1$ into the space $D_1$ centines to the space $D_0$, from the fact that it is $D_1$-continuous, it follow that its continuity under the mapping into the space $D_0$, the space $D_1$. Now it remains to apply the well-known theorem of functional analysis about the fact that a linear operator, continuously mapping one normed space into another, is bounded.

Sufficiency. Let the sequence $\eta_1, \eta_2, \ldots$ ($\eta_k \in C_l$) be $D_1$-convergent. Then there is a certain number $k_0$, $\eta_k \in D$; and $\|\eta_k\|_{D_1} \to 0$ as ($k \geqslant k_0$). From the fact that $\Omega D_1 \subseteq D_0$, it follows that the sequence $\{\Omega\eta_k\}$, having a number $k_0$, belongs to the space $D_0$. Since the operator $\Omega$ as is bounded, then $\|\Omega\eta_k\|_{D_0} \leqslant \|\Omega\| \cdot \|\eta_k\|_{D_1}$ for $k \geqslant k_0$. From this it follows
bounded, then $\|\Omega\eta_k\|_{D_0} \leqslant \|\Omega\| \cdot \|\eta_k\|_{D_1}$ for $k \geqslant k_0$. From this it follows
that $\|\Omega\eta_k\|_{D_0} \to 0$ as $k \to \infty$, $k \geqslant k_0$.

Let us return to your problem. Let us introduce in the space $V(\infty)$ a series of nour convergence.

Let us return to your problem. Let us introduce in the space $V(\infty)$ a series of convergence.

Definition 1. The sequence $\eta_1, \eta_2, \ldots$ ($\eta_k \in V(\infty)$) is called *B-convergent*, if for $k \to \infty$ uniformly in $a \in (a = 1, 2, \ldots)$ tends to zero $\sup\limits_{0<a} \|\eta_k(t)\|_{E_n}$; the same sequence is called *V-convergent*, if for $k \to$ uniformly in $a$, it tends to $y(a)\eta_k$ finally, if the sequence $\eta_1, \eta_2 \ldots$ is ν-convergent, if for $k \to \infty$ uniformly in $a$, it tends to zero $\max\limits_{0<t<a} y(t)\eta_k$.

Let us clarify the relationship of the given convergences. Obviously, $V$-convergence is stronger than ν-convergence and $B$-convergence. Further, the space $V(\infty)$, being conjugate to the space $c(0, \infty)$ of continuous finite functions with values in the space $E_1$, has the topology of weak convergence. Comparing ν-convergence and $V$-convergence with weak convergenno, it can be established that the latter is weaker than the former. For this, by virtue of the remark made above, it is sufficient to ensure the correctness of this fact for ν-convergence. Invoking the integral representation of linear continuous functionals on the space $c(0, \infty)$ taking into account the inequality for Stieltjes integrals, we will have

$$\|\eta_k(x)\|_{\substack{E_n \\ x \in c(0,\infty)}} = \| \int_0^{t(x)} x(t)\, d\,\eta_k(t)\|_{E_n} \leqslant \max_{0<t<t(s)} |x(t)| \cdot \overset{t(c)}{\underset{0}{V}} \eta_k \leqslant$$

$$\leqslant (1 + [t(x)])\|x\|_{s(0,\infty)} \cdot \max_{0<t<[t(x)]} \overset{t+1}{\underset{t}{V}} \eta_k.$$

In these estimates, $t(x)$ denotes a constant that satisfies the condition: $x(t) = 0$, if $t \geqslant t(x)$. But by assumption $\max\limits_{0<t<[t(x)]} \overset{t+1}{\underset{t}{y}}(t)\eta_k$ tends to zero as $k \to \infty$. Therefore, it tends to zero and $\eta_k(x)$, which was required to establish.

In the future, we will accordingly denote the space $V(\infty)$, endowed with $B$-convergence, $V$-convergence, ν-convergence, by $BV(\infty)$, $VV(\infty)$, $\nu V(\infty)$, $\nu V(\infty)$.

Regarding the asymptotic property of convergent sequences in only the indicated spaces is justified.

Figure 4: Figure 4

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Figure 5: Figure 5